THE

MATHEMATICAL GAZETTE

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JAMES JOSEPH SYLVESTER

JAMES JOSEPH SYLVESTER (1814–1897) made two trips to the United States : the first (Nov. 1841–Feb. 1842) when he was for a very short time professor of mathematics at the University of Virginia, and the second (Feb. 1876–Dec. 1883) when he was professor of mathematics at The Johns Hopkins University, where scholars were

" Led by soaring-genius'd Sylvester "

as the poet Lanier has expressed it in his " Ode to The Johns Hopkins University ". This University was then a source of seething inspiration in mathematical research, and Sylvester founded and edited, for several years, the *American Journal of Mathematics*. From December 1883 until his death Sylvester was Savilian professor of geometry at Oxford.

Shortly before leaving Baltimore his portrait was painted by Harper Pennington, a pupil of Whistler and a portrait painter of some renown ; it now hangs in the Gilman Memorial Room in Gilman Hall. This Hall was named after Daniel Coot Gilman, the first President of the University, and an enthusiastic admirer and intimate friend of Sylvester for the last twenty years of his life. The portrait is now published in this issue of the *Gazette* for the first time.

In 1886 Sylvester was made an Honorary Member and in 1891 the President of the Association for the Improvement of Geometrical Teaching, which afterwards became the Mathematical Association.

The thanks of the Association are due to The Johns Hopkins University for permission to publish this portrait, and to Professor R. C. Archibald for his suggestion that it should be published in the *Gazette*, for his kindness in making all arrangements to obtain a copy, and for his notes printed above.

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THE CELESTIAL CUBE.

By A. Buxton.

I. In the May number of the *Gazette* for 1943, Sir Percy Nunn contributed an excellent article on the celestial cylinder. He called it " an excursion intended to illustrate an application of elementary mathematics to astronomy ". My experience has shown that, while many people are interested in the apparent motions of the heavenly bodies, few are prepared to make the necessary effort to obtain that degree of calculation in the subject, which makes it so much more fascinating. One of the main stumbling-blocks against attaining that precision, which Sir John Herschel so strongly emphasises in his *Outlines of Astronomy*, is the lack of knowledge of spherical trigonometry. The celestial cylinder and cube have been developed, therefore, in the endeavour to convert spherical into plane trigonometry. Mercator rendered signal service in transferring maps from the sphere to the plane. This movement has continued to such an extent that, in the intense concentration on the plane, children often lose sight of the spherical nature of the earth. They appreciate the beauties of the sky, they love to watch the relative positions of the constellations,

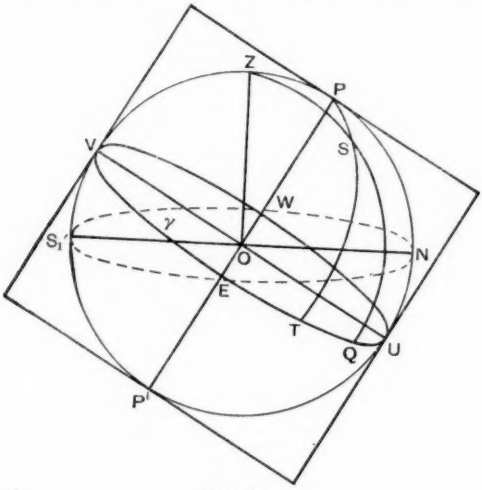


Fig. 1.

but plane star maps are much more acceptable to them than spherical. Even with older students, attempts to introduce problems on the astronomical triangle are seriously hampered by a lack of knowledge of spherical trigonometry. To maintain their interest, one is obliged somehow to convert spherical trigonometry into plane trigonometry. Sir Percy Nunn has shown how this can be done by projecting points on the surface of the celestial sphere radially outwards from the centre of the sphere on to the surface of a cylinder, which just fits over the sphere, with its axis passing through the centre of the sphere and the north pole of the heavens. Most children can recognise the constellation known as the Plough, and can obtain from it the position of the North star or Pole star. In Fig. 1, OP is the line from the observer O to the

Pole star P. The great circle with its plane perpendicular to OP is known as the celestial equator. The point Z vertically overhead is called the zenith and the vertical plane through O, Z, P is the meridian. This great circle cuts the celestial equator in the points U and V. One pair of opposite faces of the cube touches the sphere at U and V respectively, another pair at the poles P, P', and the third pair at E and W respectively, where E and W are the east and west points and are the intersections of the celestial equator with

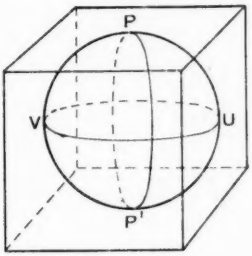


FIG. 2.

the horizon, the great circle with its plane perpendicular to OZ. The six square faces then have their centres at U, P, V, P', E, W.

II. Now if we imagine ourselves at O looking outwards to the celestial sphere, stars on it such as S will appear to be on the cube at points where lines such as OS meet one of the faces of the cube. Confining our attention, in the first place, to the square whose centre is U, let OS meet this square in S_1; then the following results are evident.

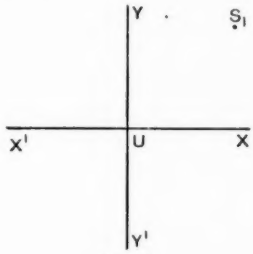


FIG. 3.

(i) If S is on the celestial equator, it will project into a point on the axis xUx'.

(ii) If S is on the meridian, it will project into a point on the axis yUy'.

(iii) If S is on a great circle through the poles P, P', it will project on to a line parallel to the y-axis.

(iv) If S is on a great circle through E, W, it will project into a point on a line parallel to the x-axis, provided, in all these cases, that OS meets the face of the square under consideration. Any great circle, therefore, projects into a line, which is the intersection of the plane of the great circle and the face of the cube under consideration.

Now taking the observer O on the surface of the earth, its north pole being P_1, a line OU_1 parallel to OP_1 points to the north (Fig. 4), that is, OU_1 is

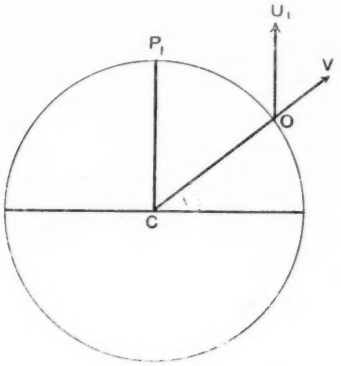


FIG. 4.

parallel to OP in the celestial sphere (Fig. 1). The radius through O produced, OV, points to the zenith, that is, in the celestial sphere OV is parallel to OZ in Fig. 1 and the angle between OP and OZ (Fig. 1) in the celestial sphere is equal to the angle between OU_1 and OV on the earth, so that

$$\angle OCP_1 = 90° - \lambda = \angle ZOP.$$

This connects the earth sphere with the celestial sphere.

III. The position of objects in the sky, such as a star S (Fig. 1), can be determined by coordinates like latitude and longitude on the surface of the earth. Since S in its diurnal motion describes a small circle in the sky about PP' as its axis, the arc ST remains constant throughout its motion. This coordinate is called declination (δ) and corresponds to latitude. Taking a sphere of unit radius, $PS = 90° - \delta$. The sun crosses the celestial equator from south to north through the first point of Aries, Υ, about 21st March. The second coordinate required for fixing the star with reference to the celestial equator is ΥT, or the angle ΥOT, which is called Right Ascension and increases towards the east. This coordinate clearly corresponds to longitude. The great circle PST is called the hour circle; the angle ZPS or the angle which the hour circle makes with the meridian is the hour angle and the hour angle and declination will also fix the position of the star. The trouble is, of course, that the hour angle varies with the time.

The most obvious system of coordinates is elevation and bearing, used in plane trigonometry (Fig. 5). Here the horizon is taken as the fundamental circle. The angle $SOH(\gamma)$ is called the altitude rather than elevation, and the second coordinate is the angle NOH, termed azimuth rather than bearing. Let us measure azimuth clockwise from the north. The disadvantage of these coordinates is that they are continually changing, whereas declination and right ascension can be regarded as constant. Thus if λ is the latitude of the observer, we have in the astronomical triangle PZS (Fig. 1):

$$PZ = 90° - \lambda, \quad SZ = 90° - \text{altitude} = 90° - \gamma, \quad SP = 90° - \delta.$$

$$\angle PZS = \text{azimuth}, \quad \angle ZPS = \text{hour angle}.$$

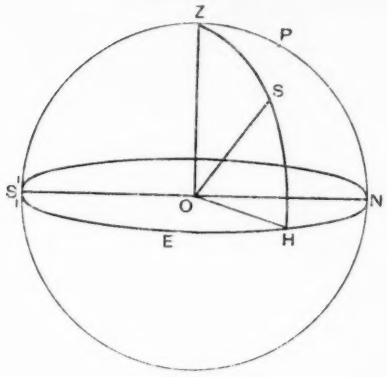


FIG. 5.

P = Pole star, NES'$_1$ = horizon.

IV. (A) If we take the face of the cube centre P, the right ascension of the meridian is ΥU, and if the right ascension of S is $\alpha(\Upsilon T)$,

$$\angle ZPS = 180° - (\Upsilon U - \Upsilon T) = 180° + \alpha - \phi,$$

where $\phi = \Upsilon U$. Let OS meet the face in S_2 (Fig. 6).

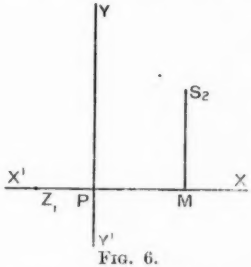


FIG. 6.

$$\angle POS = 90° - \delta = \angle POS_2,$$

$$S_2P/OP = \tan(90° - \delta) = \cot\delta,$$

and if the radius of the sphere is unity, $S_2P = \cot\delta$.

If S_2M is parallel to YY',

$$\angle S_2PM = 180° - ZPS = \phi - \alpha,$$

$$PM = \cot\delta\cos(\phi - \alpha), \quad S_2M = \cot\delta\sin(\phi - \alpha).$$

Now Z projects into Z_1 on XX' such that

$$PZ_1 = \tan(90° - \lambda) = \cot\lambda,$$

since $\angle POZ = 90° - \lambda$. The line Z_1S_2 is the projection of the great circle with its plane through ZOS (Fig. 7), that is, Z_1S_2 subtends $90° - \gamma$ at O, so that

$$OZ_1 = \operatorname{cosec} \lambda, \quad OS_2 = \operatorname{cosec} \delta,$$

and
$$Z_1S_2^2 = \cot^2 \lambda + \cot^2 \delta + 2 \cot \lambda \cot \delta \cos (\phi - \alpha)$$
$$= \operatorname{cosec}^2 \lambda + \operatorname{cosec}^2 \delta - 2 \operatorname{cosec} \lambda \operatorname{cosec} \delta \sin \gamma,$$

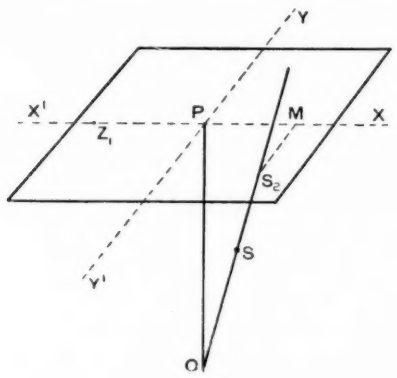


FIG. 7.

that is,
$$\cot \lambda \cot \delta \cos (\phi - \alpha) = 1 - \operatorname{cosec} \delta \operatorname{cosec} \lambda \sin \gamma,$$

or
$$\sin \gamma = \sin \lambda \sin \delta - \cos \delta \cos \lambda \cos (\phi - \alpha),$$

or, in Fig. 1, $\cos ZS = \cos PZ \cos PS + \sin PZ \sin PS \cos ZPS$.

The path of a star in the sky is a small circle around OP as axis, and this will project into a circle centre P and radius cot δ. The meridians project into radii centre P; the horizon will project into a line distant tan λ from

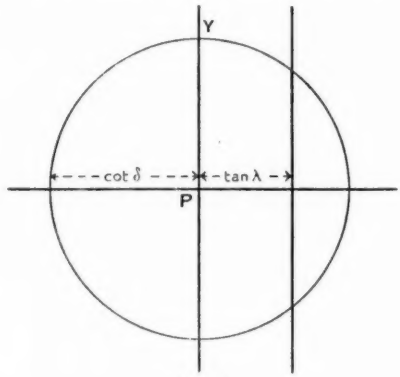
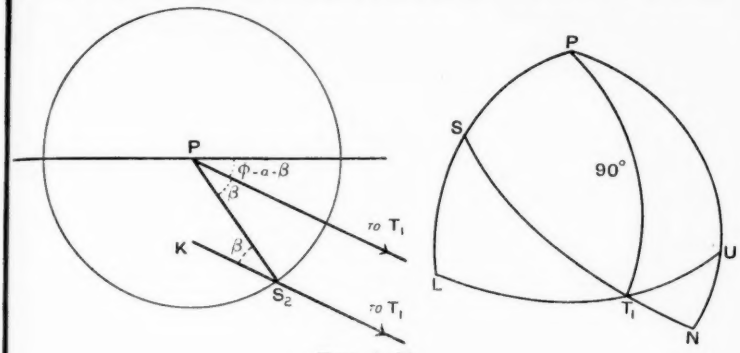


FIG. 8.

and parallel to the axis PY (Fig. 8). Rising and setting will occur if this line intersects the circle radius cot δ, that is, if tan $\lambda <$ cot δ or tan λ tan $\delta < 1$.

Stars are known to have their own proper motions, but these are small, and, for all practical cases, we may neglect them, but in the case of a star



FIGS. 9, 10.

moving over the ages in the direction S_2T_1 (Fig. 9), we should know its declination ($90° - SP$) and its direction ($\angle PST_1$) from records. Then, since $PT_1 = 90°$,

$$\tan SPT_1 = \tan LST_1 \sin \delta.*$$

The arc PS (Fig. 10) will project into the radius PS_2 (Fig. 9), the great circle ST_1N will project into S_2K and the equatorial arc LU will go to infinity, so that S_2K and the parallel through P to S_2K will meet at infinity in the projection of T_1. The angle SPT_1 will project into itself and is marked β (Fig. 9). Hence if the declination is $\delta = 26°$, and $\angle LST_1 = 29°$, then

$$\angle SPT_1 = \tan^{-1}(\tan 29° \sin 26°) = 13\tfrac{1}{2}° = LT_1 = \beta.$$

If the R.A. of the star S is 153°, this means that after centuries the star will eventually meet the equator at the equatorial point

$$T_1(153° + 13\tfrac{1}{2}° = 166\tfrac{1}{2}°).$$

Drawing in Fig. 11 the plane triangle PQS_2, with $PS_2 = \cot 26°$,

$$\angle S_2PQ = 153°, \quad \angle PS_2Q = 13\tfrac{1}{2}°,$$

PQ represents cot 26°, thus giving the proper motion intercept on the R.A. 0° circle as 26°, since S_2Q is the projection of the proper motion on the tangent plane at P. Having now the intercepts of the proper motion on the celestial equator and on the meridian through Υ, we can now repeat the intercepts for a group of stars and determine the approximate position of their apex.

(B) If, on the other hand, we take a cube with one face centre Z, then Z becomes the origin instead of P.

$$\angle P_1ZS_3 = \text{azimuth} = \psi, \text{ say,}$$

$$OP_1 = \operatorname{cosec} \lambda, \quad OS_3 = \operatorname{cosec} \gamma.$$

P_1S_3 subtends $90° - \delta$ at O, so we have to interchange δ and γ, and replace $-\cos(\phi - \alpha)$ by $\cos \psi$, giving

$$\sin \delta = \sin \lambda \sin \gamma + \cos \lambda \cos \gamma \cos \psi ;$$

* For proof of a similar formula, see Fig. 15, IV (C).

hence, if in latitude $51\frac{1}{2}°$ N. a star has azimuth 60° and altitude 37°,

$$\sin\delta = \sin 51\tfrac{1}{2}° \sin 37° + \cos 51\tfrac{1}{2}° \cos 37° \cos 60°$$
$$= 0{\cdot}7196$$

and $$\delta = 46° \text{ N.}$$

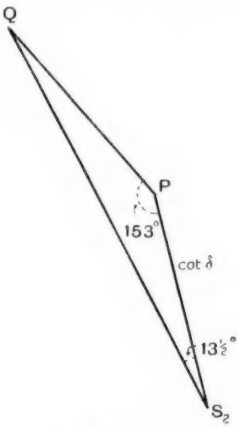


FIG. 11.

From a table of declinations of well-known stars it can be seen that the only possibility is Capella. This method can therefore be used for star identification. Repetition of observations on Capella will merely indicate that, whereas altitude and azimuth are continually changing, the formula gives a constant value of the declination.

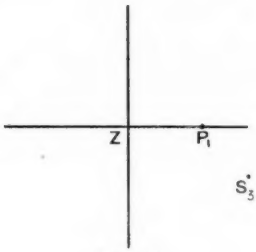


FIG. 12.

(C) If we take U as the centre of the face on to which the star S projects (S_4), then P projects to infinity along UY and Z projects into Z_2 below U on UY'. The great circle PST projects into a line through S_4 parallel to the y-axis, and so T in Fig. 1 projects into T_1 in Fig. 13.

$$UT_1 = \tan UOT = \tan(\phi - \alpha), \quad OT_1 = \sec(\phi - \alpha), \quad \angle S_4OT_1 = \delta.$$

Hence $$S_4T_1/OT_1 = \tan\delta, \quad S_4T_1 = \tan\delta \sec(\phi - \alpha),$$

$$UZ_2/OU = \tan\lambda, \quad UZ_2 = \tan\lambda.$$

If $UK = \sin(\phi - \alpha)$, $\quad UK/UT_1 = KB/S_4T_1$,

that is,
$$\frac{\sin(\phi-\alpha)}{\tan(\phi-\alpha)} = \frac{KB}{\tan\delta\sec(\phi-\alpha)},$$

or
$$KB = \tan\delta.$$

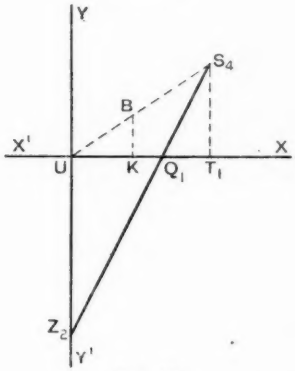


FIG. 13.

Let the great circle ZS meet the celestial equator in Q. Draw the great circle from P to Q. The arc $PQ = 90°$. Let $\angle UPQ = \phi - \beta$. Then Q projects into a point Q_1 on the x-axis (Fig. 13), between U and T_1, and the great

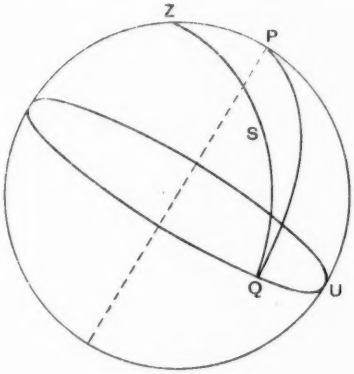


FIG. 14.

circle ZS projects into the line S_4Z_2; that is, Q_1 is the intersection of S_4Z_2 and the x-axis.

$$\tan S_4Q_1T_1 = S_1T_1/Q_1T_1 = UZ_2/UQ_1 = \tan\lambda/\tan(\phi-\beta).$$

At Q let χ be the angle which the great circle ZSQ makes with the celestial equator; then in Fig. 15 $\tan \chi = p/a$, $\tan \lambda = p/b$, and

$$\tan \chi / \tan \lambda = b/a = \operatorname{cosec} (\phi - \beta),$$

that is,

$$\tan \chi = \tan \lambda \operatorname{cosec} (\phi - \beta).$$

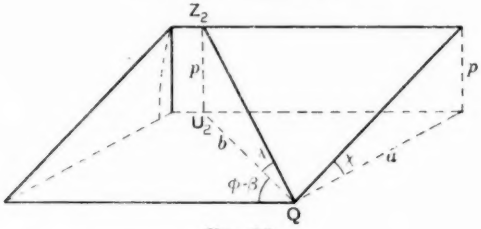


FIG. 15.

$$\tan S_4 Q_1 T_1 = \tan \chi \sin (\phi - \beta)/\tan (\phi - \beta)$$
$$= \tan \chi \cos (\phi - \beta);$$

that is, the angle χ on the sphere is greater than the angle in the plane since $\cos (\phi - \beta) < 1$. This formula enables us to transfer angles made by great circles with the celestial equator on to the plane. Declination parallels project into circles centre P on the tangent plane through P and hyperbolas on the tangent plane through U. The horizon projects into a line parallel to XUX' and distant $\cot \lambda$ from it. The equation of the hyperbola is given as the section of $x^2 + z^2 = y^2 \cot^2 \delta$ by the plane $z = 1$, namely

$$1 = (y \cot \delta)^2 - x^2,$$

so that the hyperbola cuts the y-axis at distance $\tan \delta$ from U.

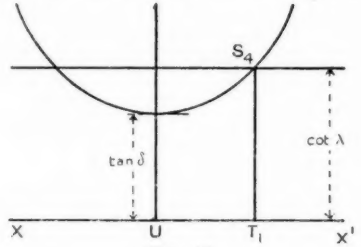


FIG. 16.

As before,

$$UT_1 = \tan (\phi - \alpha),$$

so that

$$1 = (\cot \lambda \cot \delta)^2 - \tan^2 (\phi - \alpha),$$

or

$$\sec^2 (\phi - \alpha) = (\cot \lambda \cot \delta)^2,$$

that is,

$$\cos (\phi - \alpha) = -\tan \lambda \tan \delta,$$

giving the formula for the hour angle in degrees for rising and setting. If the azimuth of the point of rising and setting is θ (Fig. 17), then since $OU = 1$ and $UW = \cot \lambda$, we have

$$\tan \theta = \tan (\phi - \alpha)/OW = \tan (\phi - \alpha) \sin \lambda.$$

Rising and setting therefore occur at $\theta°$ east and west of the north point, where $\theta = \tan^{-1}\{\tan(\phi - \alpha)\sin\lambda\}$.

The projection of the ecliptic is also a line parallel to XUX' at a height $\cot i$ above XX', where i is the inclination (23° 27′) of the ecliptic to the celestial equator. If α_1 is the right ascension of the sun, we must replace $\phi - \alpha$ by $90° - \alpha_1$, and i by $90° - \lambda$, so that

$$\sin\alpha_1 = \cot i \tan\delta,$$

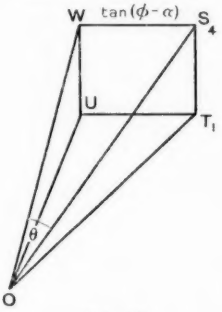


FIG. 17.

where δ is the declination of the sun, or

$$\sin\alpha_1 \tan i = \tan\delta.$$

If θ is the azimuth of the sun on the ecliptic, or the celestial longitude, then

$$\tan(90° - \theta) = \cot\alpha_1 \cos i,$$

or

$$\tan\theta = \tan\alpha_1 \sec i.$$

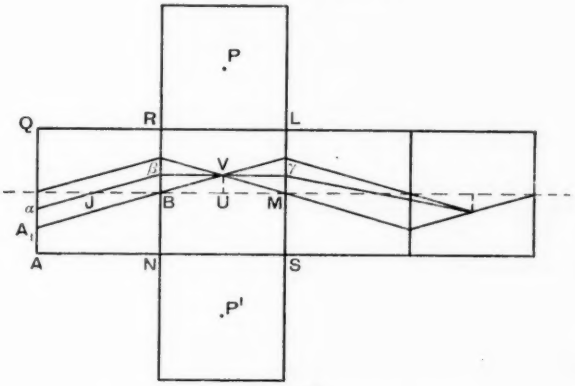


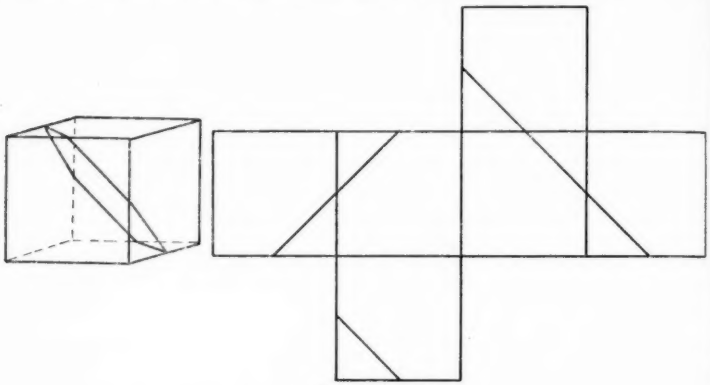
FIG. 18.

P = Pole star, dotted line = celestial equator.

(D) It remains to sketch the relations between the various squares.

(*a*) Sides QA, LS represent meridians 180° apart.

(*b*) All lines through B, the midpoint of one side RN, are unchanged in direction in the next square. This is illustrated in the regular hexagon obtained by cutting a cube by a plane so orientated that the intersections with the edges are at the midpoints of the edges.



FIGS. 19, 20.

(*c*) All lines through the centre of one square emerge parallel to the common axis in the next square. Thus $\alpha\beta$ passes through the centre J of the square $QRNA$ and thus emerges in $\beta\gamma$ parallel to BM.

(*d*) Parallel lines in one square, A_1B and $\alpha\beta$, meet in a point on the y-axis in the next square, and they become parallel again in the third square, meeting in the y-axis again in the fourth. In certain cases the planes of the squares may have to be extended indefinitely, especially in the y-direction.

(*e*) For example, to connect A and B in two adjacent squares (Fig. 21). The parallel to the axis through A passes through the centre G of the next

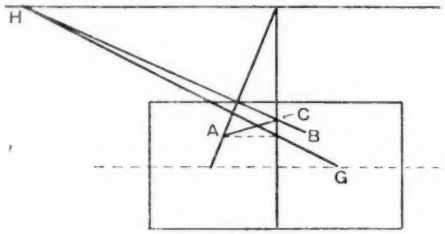


FIG. 21.

square. The line through A and the centre of its own square emerges parallel to the axis and meets the line through G in H. HB cuts the common edge of the two squares in C. The path is therefore ACB, and on the sphere the path is a great circle passing through the points of which A, C, B are the projections.

V. The plane triangle can sometimes help in the derivation of formulae in spherical trigonometry. For example, in the spherical triangle ABC (Fig. 22),

we may take A as the north pole, AB as the meridian, and AC as an hour circle. We may also project from O instead of from the centre of the sphere. Under this system of projection, circles on the sphere project into circles on

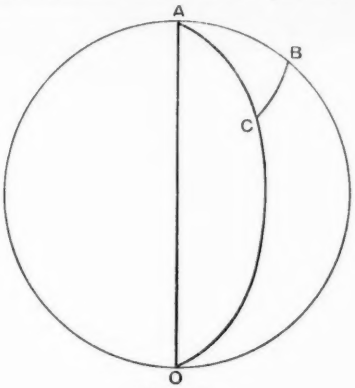


FIG. 22.

the polar plane or the tangent plane at A, and the angles on the sphere project into equal angles on the plane. The spherical triangle ABC therefore projects into $AB'C'$ (Fig. 23), where the angle ABC on the sphere is equal to the angle

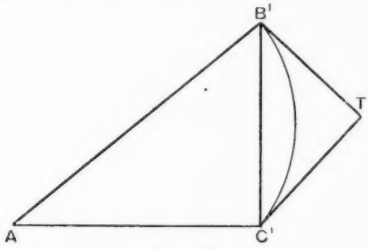


FIG. 23.

$AB'T$ on the plane, and ACB on the sphere is equal to $AC'T$ on the plane, so that the angles of the triangle $AB'C'$ are $A, B - \frac{1}{2}E, C - \frac{1}{2}E$, where E is the excess of the angles of the spherical triangle over 180°, and $\angle C'B'T = \frac{1}{2}E$.

$$AB' = 2\tan\tfrac{1}{2}c, \quad AC' = 2\tan\tfrac{1}{2}b,$$

$$OB' = 2\sec\tfrac{1}{2}c, \; OB = 2\cos\tfrac{1}{2}c, \; OC' = 2\sec\tfrac{1}{2}b, \; OC = 2\cos\tfrac{1}{2}b.$$

Now $OB' \,.\, OB = OC \,.\, OC'$, so that B, C, C', B' are concyclic and the triangles OCB, $OB'C'$ are similar (Fig. 24).

Thus $\quad B'C'/BC = OC'/OB,$

but $\quad BC = 2\sin\tfrac{1}{2}a,$

so that $\quad B'C' = 2\sin\tfrac{1}{2}a/\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c,$

and the sides of the triangle $AB'C'$ are proportional to

$$\sin\tfrac{1}{2}c\cos\tfrac{1}{2}b, \quad \sin\tfrac{1}{2}b\cos\tfrac{1}{2}c, \quad \sin\tfrac{1}{2}a.$$

The triangle ABC', where C' is diametrically opposite to C, projects into the triangle AB_2C_2 (Fig. 25). Here the new spherical excess is $2C - E$,

$$AC_2 = 2\cot\tfrac{1}{2}b, \quad AB_2 = 2\tan\tfrac{1}{2}c, \quad B_2C_2 = 2\cos\tfrac{1}{2}a/\sin\tfrac{1}{2}b\cos\tfrac{1}{2}c.$$

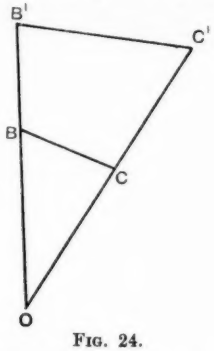


FIG. 24.

The sides are therefore proportional to $\cos\frac{1}{2}a$, $\cos\frac{1}{2}b\cos\frac{1}{2}c$, $\sin\frac{1}{2}c\sin\frac{1}{2}b$, and the angles are $\pi - A$, $C - \frac{1}{2}(2C - E)$, $\pi - B - \frac{1}{2}(2C - E)$, that is, $\pi - A$, $\frac{1}{2}E$, $A - \frac{1}{2}E$. Thus,

$$\begin{aligned} 2s' &= \cos\tfrac{1}{2}a + \cos\tfrac{1}{2}b\cos\tfrac{1}{2}c + \sin\tfrac{1}{2}c\sin\tfrac{1}{2}b \\ &= \cos\tfrac{1}{2}a + \cos\tfrac{1}{2}(b-c) \\ &= 2\cos\tfrac{1}{4}(a+b-c)\cos\tfrac{1}{4}(a-b+c), \end{aligned}$$

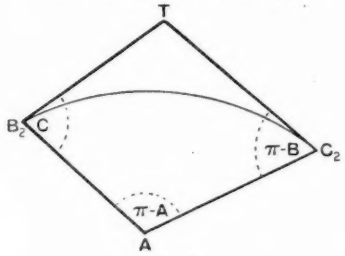


FIG. 25.

and

$$s' = \cos\tfrac{1}{2}(s-c)\cos\tfrac{1}{2}(s-b).$$

$$\begin{aligned} 2s' - 2a' &= \cos\tfrac{1}{2}(b-c) - \cos\tfrac{1}{2}a \\ &= 2\sin\tfrac{1}{4}(b-c+a)\sin\tfrac{1}{4}(a-b+c), \end{aligned}$$

$$s' - a' = \sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-b).$$

Similarly,

$$s' - b' = \sin\tfrac{1}{2}s\sin\tfrac{1}{2}(s-a),$$

$$s' - c' = \cos\tfrac{1}{2}s\cos\tfrac{1}{2}(s-a).$$

$$\begin{aligned} \tan\tfrac{1}{4}E &= \sqrt{\left\{\frac{(s'-b')(s'-a')}{s'(s'-c')}\right\}} \\ &= \sqrt{\left\{\frac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-a)\cos\frac{1}{2}(s-b)\cos\frac{1}{2}(s-c)}\right\}} \\ &= \sqrt{\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}}. \end{aligned}$$

If we take the origin at O and OA as the axis of z, the equation of the sphere is, with unit radius,

$$x^2+y^2+z^2=2x;$$

this is cut by the plane $ax+by+cz=1$ where

$$x^2+y^2+z^2=2x(ax+by+cz).$$

This cone cuts the plane $x=2$ where

$$y^2+z^2=4(2a+by+cz)-4.$$

This is a circle,* so that the projection of any circle on the sphere is a circle.

The angle at $V=\angle UVT$ (Fig. 26); tangents to the sphere from U or from T are equal, so that $UV=UO$, $TV=TO$, and UT is common. Thus the triangles TVU, TOU are congruent, and hence

$$\angle TOU=\angle UVT,$$

that is, the angle on the sphere projects into an equal angle on the face of the cube centre P (Fig. 1) provided that the centre of projection is P'.

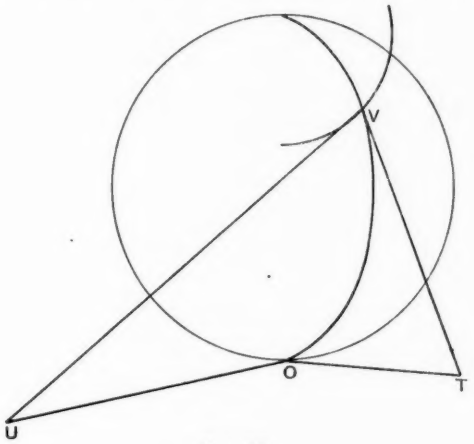


FIG. 26.

The amateur in astronomy finds added zest in being able to solve problems on rising and setting, altitudes and azimuths, and so on, by calculation. He will have seen now that lack of knowledge of the sphere need not act as a deterrent, and that his knowledge of school mathematics will serve his purpose. Having taken so much trouble to avoid spherical trigonometry, he will then wisely make the extra effort to become familiar with the methods of spherical trigonometry, starting with the spherical cosine rule, the sine formula and the four-part cotangent rule. A. BUXTON.

* The centre is $(2, 2b, 2c)$. The line joining the centre to O is $x/1=y/b=z/c$, which passes through the vertex of the tangent cone to the plane section. The coordinates of the vertex are $1/(1-a)$, $b/(1-a)$, $c/(1-a)$.

MATHEMATICS IN SCHOOL EXAMINATIONS, 1950.*

Mr. W. J. Langford (Battersea Grammar School) : The purpose of the discussion this morning is two-fold : it is hoped that it will provide an opportunity, now overdue by nearly 10 years, for the members of the Association to comment upon, to argue and perhaps to complain about, the content of mathematical syllabuses in schools—I hope also that the discussion will enable the *ad hoc* Committee appointed by the Council to embark without delay upon the task of preparing a report for the Ministry of Education upon this problem of teaching and examining Mathematics in Secondary Schools. The publication of the Norwood Report made us think seriously about this matter, but the war intervened and nothing was done. The recent Report on Examinations in Secondary Schools produced by the Secondary Schools Examination Council has caused us all to consider the general problem once again. My part in this discussion is to give a brief summary of the Report as it affects our own subject, and then to make some comment upon the spirit and content of the mathematical syllabus in Secondary Grammar Schools. I should have been followed by Mr. M. W. Brown, a former colleague, at present a lecturer at Wandsworth Emergency Training College ; Mr. A. W. Riley will be reading his paper, which deals with the work in Secondary Modern Schools and will make some reference to teaching problems in Primary Schools. Miss F. M. Pendry, Senior Mathematics Mistress at James Howard School, will comment upon what we have said, stating the special claims of the Girls' Schools, and Dr. E. A. Maxwell, of Queens' College, Cambridge, will discuss the possible effects upon the examinations themselves. I would add that it must not be assumed that the opinions of any one of us are necessarily held, even in part, by any other, and Mr. Riley reserves the right to criticise the opinions he puts before you on behalf of Mr. Brown.

I say at once that I am in no way alarmed by the present proposals of the Secondary Schools Examination Council. As I see it, we are now given a real opportunity to say emphatically and precisely what part we wish our subject to play in the new scheme of things. The proposals as they stand convey a real freedom of action to schools capable of profiting by it and willing to experiment. In any sensible academic progress there must be hurdles which will indicate stages of achievement as the pupil flies them (our own internal tests are precisely of this kind) but they really *should* be taken in the normal stride. All too often they are immense barriers which leave the child defeated or exhausted at the very point where he should be reaching out enthusiastically towards some understanding and appreciation of the subject, worth-while as a study for its own sake. Within our mathematical departments we now have the opportunity to say in advance what the scope of the work should be, and, more important still, we have the chance to plan our work according to the needs of our pupils. I am going to say that we should hold the view that mathematics ought not to be taught *as an examination subject* to every child right through the Lower and Middle School ; I am, nevertheless, going to say that the subject should be an essential element in the general education of every child up to the age of about 16.

Clearly we shall expect the Universities and professional bodies to lay down conditions concerning exemption requirements, and here too I believe we must make our position plain. At present, under the regulations of one University, Mathematics is required to full credit standard in the normal scheme of exemption from matriculation. But there are two other schemes of exemption : in one a reasonably good pass in Mathematics will do and in the other

* A discussion at the Annual Meeting of the Mathematical Association, January 3rd, 1948, the President, Professor G. B. Jeffery, in the chair.

Mathematics can be omitted altogether. We should state a case now for Mathematics to be required at the Ordinary level in the new General Certificate of Education if the student concerned intends to proceed to further study involving Mathematics, Engineering or Science (including Geography, Economics and Commerce). Equally firmly we should say that it ought not to be *required* as a constituent in an exemption prior to further study on the Arts side ; it is just for such pupils as this that we may want to use our time differently in the Middle School.

All that I need say here about the general recommendations in the Report is that the examination is to be taken not earlier than the age of 15·7, that there is to be no *upper* limit of age, and that papers will be set at three levels, Ordinary, Advanced, and Scholarship. There are to be no group requirements of any kind so that apparently normal proficiency in our own language is not regarded as an essential constituent of a general education at the grammar school level. It is recommended that the new scheme should operate in 1950 : in my view this is at least a year too soon for associations like our own to discuss the proposals thoroughly, to report conclusions, and still allow reasonable notice to be given of the major changes which will take place. My last point of comment concerns the fact that the Report was unanimous : personally, I am uneasy about this and I cannot rid myself of the feeling that the wording in some places is such that it meant one thing to some members of the Council and something vastly different to others. This will recoil upon us if we attempt to secure any elucidation of the many plausible generalities which compose some sections of the Report. Nevertheless I repeat that, if the recommendations mean what I want them to mean, the grammar schools should use the Report freely as an instrument of release from the so-called tyranny of the external examination.

We may now perhaps turn to the specific problem of the content of the mathematical syllabus throughout the normal 7 years of the child's grammar school career : this period is of course reduced to 5 years in the case of many, since they leave at the age of 16. We must recognise that in Mathematics we have a particularly difficult scheme to consider and it is certain that one set of papers—Ordinary, Advanced and Scholarship—cannot meet our needs, even allowing Mechanics to rank as a separate subject with a full set of papers of its own. We must consider carefully the need for retaining something equivalent to Advanced Mathematics at the present School Certificate level. For my part I should be content with papers in Mathematics, Mechanics at the Ordinary level, together with three sets at the Advanced level, one for the specialist, another for the scientist (some of these to include Mechanics as an essential part of Mathematics) and possibly a third set in Pure Mathematics alone for the Sixth Former on the Arts side who wishes to continue the subject beyond the Ordinary level. At the Scholarship level only two sets should be necessary, one for the specialist and the other for the scientist.

If as an Association we agree with the recommendations of the Schools Examination Council and support the proposal that there should be this single examination with papers at the three levels, we should then go forward at once with the task of overhauling completely our existing syllabuses so that we may be able to present the various examining bodies with our ideas on the content of the examination ; this Association is surely the only body in this country able to speak with authority on the syllabus for these proposed examinations. If we can do this at once our position will be immensely strengthened ; it is one thing to be invited to work in consultation, it is quite another thing to take the initiative and to present a scheme in advance.

It would be out of place for me to attempt any discussion here of the detail of such a syllabus ; what I propose to do is to put before you some general

principles. We should, I believe, base our scheme on the work of the full 7 years—though it is our responsibility to safeguard the mathematical education of those who can stay only for 5 years. The 7 years should, in my view, fall into three well-defined parts, *A*, *B* and *C*. For the able boy it is best to allocate 2 years to *A*, 2 years to *B* and 3 years to *C*. For the less able it may be better to allocate 3 years to *A*, 2 years to *B* and only 2 years to *C*. For some of those who leave school at 16, or do not continue their mathematical education into the Sixth Form, we should be prepared to use the 2 years of stage *B* in an entirely new way. In stage *A* we are called upon to teach fundamental processes and the syllabus should, in my opinion, be broadly speaking the same for all children—boys or girls—in grammar schools. During this period of 2 or 3 years, all the fundamental processes of arithmetic must be taught, and here I include the ability to use tables of squares, square roots, logarithms, sines and tangents. Examples must all be simple; designed merely to impress the method and inculcate accuracy—long and involved problems must be rigidly excluded. Much the same approach should be used in algebra. The fundamental methods (solution of equations, factors, simplifications) must be taught as routine processes and there must be a thorough introduction to graphs. Formulae must receive considerable attention both for their own sake and also to provide an elementary approach to the idea of functional dependence. Again the examples must be simple; we are learning to handle the tools of a great artistic craft. In geometry I want to see all the fundamental processes brought within reach of the pupil. Theorems should be demonstrated but no attempt should be made to teach them. With the results on parallels, congruence and similarity all assumed intuitively, and those on areas and the circle revealed as they are required, the child should be taught to solve really easy riders and to cover in this way the whole of the present syllabus of School Certificate geometry. The usual constructions, scale drawing, easy plan and elevation, and freehand sketches of solid objects all have an important place in this work. Elementary mensuration comes naturally here, coordinating the work in arithmetic and geometry. Lastly, I want to see elementary trigonometry included at this stage for all pupils; again only enough to teach the fundamental process of solving a right-angled triangle by the use of the sine and the tangent functions, but sufficient to allow the child to appreciate the power of the methods. We cannot justify the present state of affairs which makes it possible to solve a 60°, 30°, 90° triangle using Pythagoras' theorem, but leaves the pupil with no means at all of handling one with angles of 55°, 35°, 90°. Such is my suggestion for the foundation stage *A* which should be taken by all, and should occupy 2 or 3 years.

In stage *B* we begin to teach mathematics in the true sense. We must now face the fact that there are several groups to be considered:

(*a*) those who will take Mathematics at the Advanced or Scholarship standard;
(*b*) those who will take it at the Ordinary standard and will leave school or, if they remain, will not read mathematics in the Sixth Form;
(*c*) those who should not take it as an examination subject at all.

We must be ready to work out suitable syllabuses, which will be essentially distinct, for these pupils. I appreciate readily the difficulties which face the smaller one-stream schools in dealing with the teaching problem thus created, but I am still certain that everything possible must be done to give the pupil the type of mathematical education he needs at this stage, and not to be content to impose one stereotyped course upon them all. I do, however, urge that, throughout the 2 years of this stage *B* work, ample opportunity should

be found to include an account of the outstanding personalities and developments in the history of mathematics.

For those who will take the subject at the Ordinary level, but not beyond, I am convinced that the spirit and scope of the Cambridge Syndicate's Jeffery Report of 1944 on School Certificate mathematics is essentially right; the emphasis is all the while upon the full use of the mathematical processes learnt in stage *A*. We should, in my view, press for the abolition in its entirety at this stage of everything that can be regarded as "bookwork". Our mathematics here is still a process: it has not yet become a study. This admirable scheme contains the elements of the calculus and sufficient extensions of graphs to break ground in the elementary parts of analytical geometry. All the work is approached from the standpoint which insists on the unification of the subject, both in its teaching and in the scope of the examination itself.

For those who will take mathematics to the Advanced or Scholarship stages we shall probably find less need to make drastic changes in our present schemes. I would commend to your earnest consideration the Cambridge Report and Syllabus on Sixth Form mathematics, published in 1945. These two Cambridge reports seem to me to give the right guiding principles in the choice of a syllabus for the 4 or 5 years covered by stages *B* and *C*. In comment, all I have time now to say is that pure mathematics and mechanics are regarded together as one subject throughout and that it should not be possible at this stage for a pupil to take one without the other, apart possibly from the Sixth Former on the Arts side who wishes to continue with pure mathematics. In pure geometry I believe that the time has come for us to make a definite stand: more time must be given to this work in the Sixth, greater scope must be afforded in the examination, and the syllabus must omit all the "dead-end" theorems which lead only to specific types of rider. The real need is to include a worth-while course in projective geometry; I cannot agree that a boy should spend 7 years on metrical geometry and be denied any acquaintance with the power of the projective form.

I have now to mention one remaining group—those who should not take an examination in mathematics even at the Ordinary level. Here I want to say that I hope we shall do two things; the first is to prevent these pupils from being forced, either by parents or by the schools, to take the examination, and the second is to insist equally firmly that the child's mathematical education shall nevertheless continue throughout the whole of the 2 years of stage *B*—though no doubt with a smaller allocation of teaching periods. I am sure that we all know full well the type of work we should like to do here, free from any claims of an examination. I suggest that we now have an opportunity. We must stand firmly by our conviction and insist upon the problem being handled in the right way.

It is here that what I have to say becomes closely linked with the part allocated to Mr. Brown, and as I give place to Mr. Riley, who will read his paper for him, I trust that what I have been able to say will lead to a lively and fruitful discussion.

Mr. M. W. Brown (Wandsworth Emergency Training College): My part in this discussion is to attempt to place before you some ideas on the effect of the Report on Mathematics in the Secondary Modern (S.M.) School. At first sight one might feel that this Report has comparatively little to do with such schools, concerned as it is largely with external examinations, but I believe that it will ultimately have a definite effect upon the teaching of Mathematics in the S.M. school.

The proposals of the Report, particularly with regard to the age at which the first external examination can be taken and the present school leaving age,

are such that it will not be possible for the S.M. School to enter its pupils for the examination, and the Council clearly regard it as unsuitable that they should be allowed to do so. Most people would, without further thought, agree with this ; but I think it will be worth while, if you will allow me to do so, to spend a few minutes in considering in a little more detail the reasons for this opinion, since it is only by such consideration that the resultant effect upon the curriculum can be properly estimated. To do this it follows that we must determine the broad principles upon which the Four Year Course—not three years and an extra year—in these schools must be planned.

As you know, the children in the S.M. Schools form some 80 per cent. of the child population in that age range, and they can broadly be thought of as those who, at the age of 11, are not yet ready to turn from the concrete to the abstract. Some of them will never be able to do so ; others will develop later, and for the best of them there exists the possibility of transfer at the age of 13. It is essential, therefore, that the avenue of approach to these children should be of a practical nature and as far as possible within their own experience. While the more academically-minded child can be interested in work of a purely intellectual character, the interest of the less gifted child can only be aroused by making the work as concrete as possible ; further, unless that interest is aroused, real progress with these children is not possible. The keynote of the curriculum in the S.M. school should, therefore, be one of activities—of " making and doing ".

Such a curriculum is not suitable for the orthodox external written examination. Further, the problems of each S.M. school are likely to be more individual than those of the grammar schools and the adoption of a ready-made examination would greatly hinder the evolution of the type of curriculum best suited to each individual school. This point of view is admirably summed up in the recent Report of the Scottish Advisory Council on Secondary Education. They said : " Any attempt to institute for pupils of 15 a certificate based on a national, or, indeed, on any form of external test would, in our opinion, be calamitous in its effects on the short courses and foredoomed to failure. It would arrest the healthy development from the bookish and verbal to the practical and realistic just when that change is most necessary and would tend to a sterile uniformity instead of the free experiment and spirit of adventure we so greatly desire."

It is clear, therefore, that each S.M. school is intended to work out its own salvation in a fashion appropriate to the particular conditions and environment in which the school is placed. The onus of developing a suitable Mathematics syllabus will therefore rest on those who teach Mathematics in each individual school. This is a grave responsibility and one which requires careful thought. As I have said, the work of the S.M. school should be approached in a practical fashion. Unfortunately, the terms " practical and realistic " have become something of a catch-phrase, and it would often appear to be implied that it is the sole basis on which to frame a Mathematics syllabus. If by " practical and realistic " we mean that we should teach only that Mathematics which will be of use to the ordinary citizen in his daily life, very little examination is needed to convince one that the average man in the street requires little or no Mathematics, and it is certain that no worth-while syllabus can be framed on this basis. I believe that before attempting to design a Mathematics syllabus for any kind of school, it is essential to have some clearly defined aims as to why we teach Mathematics. When those aims have been defined, we can begin to frame our syllabus in such a way as to give those aims their best effect. Once we have decided what the syllabus should contain we can consider how our methods of dealing with the various topics can best be made practical and realistic. In other words, the phrase " practical and

realistic " is a qualification and not the basis for designing a Mathematics syllabus, and this is as true of the Secondary Modern school as it is of the grammar school.

I cannot resist including a few remarks on the possibility of an integrated syllabus in Mathematics, since the desirability of such a syllabus in a S.M. school is clear. But in my view it is no easy task to carry into practice, or perhaps I should say, to put on paper. It seems to me that one of the big stumbling-blocks is the unavoidable necessity for drill even in the simplest processes. One may use various branches of Mathematics in one problem, but before this can be done the child has had to acquire some facility, even if only of a very simple kind, in the use of the weapons he employs. I believe that the teaching of Mathematics as a whole does not rest with any stated syllabus on paper, but in the mind and action of the teacher. The broader his outlook and the wider his knowledge of simple applications of Mathematics, the more easily will he be able to seize every opportunity of linking together in the minds of his pupils the various branches, leading them by degrees to realise that Mathematics is a whole and indivisible ; but such realisation is one of slow growth and cannot be achieved from the word " go ".

Now such teaching, though it implies wide knowledge on the part of the teacher, does not necessarily, I think, mean that he must have a high degree of technical ability. This is an important matter in the case of the S.M. school, since the specialist teacher with high academic qualifications will be rare ; the teaching of Mathematics in these schools will be largely in the hands of the non-specialist. I have no doubt that such non-specialists can, with help and guidance, acquire a sufficiently wide mathematical education of a comparatively simple kind to enable them to teach on enlightened lines. If I may be permitted to intrude a personal note, we have found that our students at an E.T.C. who specialise in Mathematics, many of them with a slender or remote background of academic knowledge can, even in the limited period of a year, be given a training which will enable them to at least begin their work as Mathematics teachers in a sound fashion. I do not pretend, of course, that they have acquired much technical skill, but they go to the schools with enthusiasm for the subject and a broad general picture of mathematical thought which should stand them in good stead. Ultimately a considerable proportion of the teaching of Mathematics in the S.M. school will lie in the hands of these men. At present many of them are Junior Members of this Association and I feel that we should make it one of our responsibilities to give them all the help and guidance possible. At present we are comparatively little known outside the bounds of the grammar schools. One way of extending our influence is through these men who, if we give them the help they need, will spread our gospel abroad. It is certain that if we do not give them this help we shall quickly lose them and a great opportunity which is unlikely to recur.

The development of a suitable curriculum in the S.M. school is intimately connected with the question of textbooks. In view of the fact that the non-specialist will largely be responsible for the teaching of the subject the available textbooks will undoubtedly have a considerable influence upon the way in which the subject is taught. Now, whether you feel that external examinations are a good or a bad thing, it cannot be denied that they have, at least in the past, had a noticeable effect on textbooks. Over and over again one can read in the prefaces some such phrase as " covers the requirements of the syllabuses for such and such examinations ". In some of these books this has meant that their contents have in fact covered those requirements only. In so far as the examination syllabus has been an enlightened one, these books may be good ; but much of the mathematics we should teach is not easily examinable and often does not appear in the text. Professor Whitehead in his

essay on " Aims of Education " has some interesting remarks to make in this respect. He said : " Whenever a textbook of real educational worth is written, you may be quite certain that some reviewer will say that it will be very difficult to teach from. Of course it will be difficult to teach from it. If it were easy, the book ought to be burned ; for it cannot be educational. In education, as elsewhere, the broad primrose path leads to a nasty place. This evil path is represented by a book or a set of lectures which will practically enable the student to learn by heart all the questions likely to be asked at the next external examination." The complete condemnation of all books from which it is easy to teach is perhaps rather sweeping, but I am certain that many of us are consciously or sub-consciously influenced in our assessment of a book by examination requirements. I know at least that I have been guilty in this respect. A common remark when one picks up a book for a quick inspection—a remark which comes almost of its own volition—is likely to be " A good book, but not easy to teach from ". I am afraid that what we often mean is that the book does not present, as Whitehead says, " the broad primrose path to the examination ". Now what do we really want in a textbook? It depends, I think, upon whether one is a specialist or non-specialist teacher. For the experienced specialist teacher, a collection of examples may suffice ; the type of book most frequently to be found in the S.M. school, often for reasons of economy, is of this kind. For the non-specialist teacher, I think something more is required, since we must remember that he will not necessarily have the breadth of knowledge to make his subject alive in spite of the textbook. It appears to me that there is a distinct need for books which will help the non-specialist to plan his curriculum, provide him with plenty of good and realistic examples and at the same time guide him in his method of approach to the various topics. Such methods should not only be logically sound but their arrangement should be adapted to the mind and ability of the pupil. In comparing books planned on this psychological approach with those which are designed only for examination purposes the late Professor Godfrey said : " A book of the former type is not useful for examination revision and this consideration is responsible for varying degrees of compromise between the two types of book." In the case of the S.M. school no such compromise will be necessary and there is already evidence of this in some of the books now available for use in these schools. I am convinced that good books of this kind are a matter of prime importance in the development of sound teaching technique and an enlightened mathematics syllabus in all schools, and particularly so in the case of the S.M. school, with its non-specialist teachers.

Before I conclude I should like to say just a few words about any possible effect of the Report upon the primary school. I think myself that in time there will be such an effect. It has certainly been true in the past that the standard and syllabus of the examination at 11 + have been influenced by the requirements of the grammar schools, which in turn have been dependent upon the period which has elapsed before the child has normally taken School Certificate. That period will now usually be longer and I hope that this will mean that there will be less pressure on the children in the primary school. Now it is undoubtedly true that in many such schools the work of the last year is dominated by the selective examination at 11 + . If the ground to be covered is less there will be a closer approach to the ideal of the children taking the examination in their stride and there will be a surer foundation. On this point the report of the A.T.T.C.D.E. on Arithmetic in the primary school makes an interesting comment as follows : " A frequent experience in a grammar school is that primary scholars are too advanced in the technique of getting certain types of sums correct, yet have little intelligent understanding of fundamental arithmetical processes." This, I am sure, is something

that is within the experience of many of us. Less pressure in the primary school should allow children to obtain a better grasp of these fundamental principles and should also give more time for other aspects of Mathematics which are often at present neglected—for instance, Spatial Knowledge, a subject on which the Report I have quoted above has two very interesting chapters ; indeed, the Report as a whole is to be strongly recommended and has much of interest for teachers in grammar schools.

I am afraid that my remarks have been very general and somewhat discursive but I hope not altogether irrelevant. The field for speculation opened by the Report is a wide one and I have been able to take up only a few of the more obvious points. Many more must have occurred to others here and I hope that they will raise them when the general discussion begins.

Miss F. M. Pendry (County Secondary School, Clapton) : When I was asked to speak at this Conference I consulted a number of mathematical teachers in girls' grammar schools. Most of them anticipate the type of organisation which Mr. Langford has put before us. I shall therefore comment first on what he has said and come to the general points later.

Three Stream Entry.

	A.	*B*.	*C*.
Years : 1 and 2.	Basic mathematics for all. Arithmetic, Algebra, Geometry, Mensuration, Trigonometry.		
	A and B.		*C*.
3.	*O* syllabus.		*O* essentials to admit of transfer to groups *A* or *B*.
4.	*O* syllabus.		Modern school mathematics, 3 or 4 periods per week.
5.	*O* syllabus.		Drop or continue the above with fewer periods.
6 and 7.	Some other institution's first year.		Possibly some mathematics for non-specialists.

In a school with a three stream entry all would take a course lasting for at least 2 years, in which the foundations of mathematical education would be laid by the simple treatment of Arithmetic, Algebra, Geometry, Mensuration, and Trigonometry.

In my opinion these are the most important years when good habits of work should be inculcated. Much of the complicated later work in Algebra could be eliminated if great attention were paid to the work of these two years. When in my Sixth Form three girls were asked to work an example from a Higher Certificate Paper, on account of elementary errors in manipulation, not one of them arrived at the type of equation which the examiner wished to test.

In the third year portions of the *O* syllabus would probably have to be taught to all pupils, though treated differently, so that a transfer from one group to another would be possible.

In the fourth and fifth years the *A* and *B* streams would take the *O* syllabus, while the *C* stream would break away from this syllabus completely.

From this table it is obvious that we shall expect less mathematics, and some mathematics of a different kind to be taken in grammar schools as time goes on.

Examination of the situation as a whole.

As far as mathematics is concerned, boys are on much firmer ground than girls.

(*a*) In the past mathematics has enjoyed a privileged position in the curriculum. Is this position artificial or is it really beneficial in a girls' school?

Mathematics is said to provide a good mental discipline. I believe this to be true and therefore I think that all pupils should take it for the first three years.

(*b*) Mathematics as a compulsory subject for entrance to the University will probably not continue. This compulsion has not been genuine for some time—a General School Certificate + Higher Certificate has given a Matriculation exemption without a credit in mathematics.

(*c*) In a girls' school the greatest enemy of mathematical teaching is the crowded curriculum. In schools carrying a large number of subjects, the virtues of Mathematics will be weighed against those of Domestic Science, Art, Music, and all the other attractive subjects which are taught today. Particularly in small schools and in schools where there is a shortage of qualified mathematical teachers it will be difficult to provide suitable courses.

(*d*) Employers and parents have had a false idea of the value of a Matriculation Certificate. Employers are still to have access to records and certificates; if they require mathematics pupils will still demand it.

(*e*) The mathematical specialist and the science specialist will require mathematics.

(*f*) The raising of the school leaving age and the extension of further education should have an influence on grammar school children even when no University course is in view and this, I believe, will help to foster academic studies and therefore mathematics.

Summary.

If, therefore, we believe in mathematics as a major subject in the curriculum we must devise syllabuses suitable to all abilities—the less rigid framework is welcome.

We should have

(1) an academic treatment of the subject for the gifted, for the scientist and for the mathematical specialist;

(2) an extension of what might be called modern school mathematics including basic mathematics for at least three years, an examination of how things work, how things are made, mathematics of citizenship, projects, correlation with other subjects, and in fact wherever possible topics of practical interest.

Dr. E. A. Maxwell (Queens' College, Cambridge): The recent report of the Secondary School Examinations Council envisages new forms for the examinations to be held during the summer of 1950. Most people agree that schools should have at least two years' notice of changes in regulations, so that the syllabuses will have to be in print by September 1948. The final drafts must therefore be settled by about June. When we recall, on the one hand, that the Report is as yet only provisional, and, on the other, that it will in any case give only the broad strategic plan on which detailed regulations must be based, we can see how necessary it is that Associations such as ours should be ready to help, in every possible way, those on whom the responsibility for the final preparations will fall.

It is, I think, true to say that relations between teachers and examiners have been improving steadily for years, and that any move to strengthen cooperation will surely be welcomed by both sets of partners. Whether a form of union which threatens to abolish the examiner at School Certificate level is a natural corollary, I am not so sure. However, we are not there yet, and perhaps we never shall be, so let us collaborate while we may.

At present, as you all know, the examinations in Mathematics can be taken at levels conveniently described by the words "School Certificate" (with "Additional Mathematics" as an optional extra), "Subsidiary", "Higher Certificate" and what I may broadly call the "Distinction" or "Scholar-

ship " level. In addition, the higher levels can themselves be subdivided into " mathematical " mathematics, and mathematics regarded as auxiliary to a course in science. In all this, there is a variety and flexibility, born of experience, which (though not perfect—what is?) should not readily be discarded.

In order to assess the new proposals, I propose to re-write them in terms of our own subject, mathematics. I hope that in doing so I shall give a fair and faithful transcription ; such is certainly my aim.

21 (*a*). *Papers shall be available of three types or levels*—" *Ordinary mathematics* ", " *Advanced mathematics* " and " *Scholarship mathematics* ".

(*b*) *The* " *Ordinary* " *papers shall be designed to provide a reasonable test for pupils who have taken mathematics as part of a wide and general course up to the age of at least* 16, *or for pupils who have taken the subject in a non-specialist way in the Sixth Form.*

I imagine that the existing School Certificate papers would, with minor changes, be suitable for the " Ordinary " level. Whether they, or any other, could *also* be used " in a non-specialist way " is very doubtful.

(*c*) *The* " *Advanced* " *papers shall be designed to provide a reasonable test in mathematics for pupils who have taken it as a specialist subject for two years of* " *Sixth Form* " *study.*

Presumably an approximation to our present Higher Certificate.

(*d*) *The* " *Scholarship* " *papers shall be designed with ample choice of questions, not necessarily covering a substantially wider field of study than those in the* " *Advanced* " *papers, to give specially-gifted pupils an opportunity for showing distinctive merit and promise.*

Presumably an approximation to our present Distinction or University Scholarship papers, only less so!

28. *As regards the* " *Ordinary* " *papers* [*in all subjects*] *we have proposed that they shall be optional and that a pass in, say, mathematics should represent competence after a course lasting till at least* 16. *We should expect that the appropriate standard would in due course approximate to what has been a* " *Credit* " *standard in School Certificate.*

This last point means, presumably, a 50 per cent. " plough " unless the quality of the candidates rises to reduce it.

29. *A pass at* " *Advanced* " *level should approximate closely to what has been the* " *Pass* " *standard at the Higher School Certificate examination.*

32. *It would be contrary to the intentions of the system for any pupil to sit successively for* " *Ordinary mathematics* ", " *Advanced mathematics* " *and* " *Scholarship mathematics* ".

This last paragraph presumably means *either* (but probably not) that a Scholarship candidate could take the three levels at once, *or* (more probably) that he should omit altogether the " Ordinary " level examination at the age of 16, with the result that his first external test in mathematics would be the crucial " Scholarship " one. (He may, of course, have taken other subjects earlier.) This seems to cast overboard a very useful barometer. Again, what happens to a pupil who omits " Ordinary " mathematics and fails in " Advanced "? After all, every examination leaves a trail of failures lagging behind it. To call mild failure in " Advanced " a pass in " Ordinary " is hardly the ideal solution, though this is the suggestion made in the Report.

I think enough has been said for you to have a broad idea of the new proposals. At the risk of repeating the earlier speakers, I should like to emphasise one or two general points about which members of the Association should be very clear. Above all others stands the fact (§ 23) that " the mere possession of a Certificate will have virtually no significance ; it will be the contents of it which provide positive information about the holder ". As this

sentence seems to me to embody what, unless averted, may be one of the greatest tragedies of modern education, I give in detail some fundamental issues for your consideration.

It was about ten years ago, I suppose, that I began to get involved " inside Education ", if I may use the phrase, and I found almost at once that there were two major troubles in the path of the schoolmaster—the pressure towards specialisation, and the multiplicity of examination requirements. Everyone was complaining that his best pupils had too much to do, leaving too little time for their general education. In addition to this load on the individual pupils, each school as a whole was burdened by the varying requirements of the different universities and professional bodies. These difficulties were widely recognised, and I think it is true to say that two or three years ago many, perhaps most, of the people who dealt with them were ready to receive proposals for reform. The so-called Jeffery Committee had paved the way for a common understanding in School Certificate mathematics, and the work of the Cambridge Committees was offering a common ground where Higher Certificate requirements in mathematics as well as in science could find meeting-places. I believe that bold proposals, imaginatively framed, might have found a quite unexpected measure of agreement, though obviously absolute unification would not have been possible or, in my opinion, desirable.

Cutting across all this, we are offered a *General Certificate of Education*, the possession of which in itself, is meaningless; one " scrape " at the Ordinary level gets it. Obviously the universities and the professional bodies will insist on special requirements to suit their own needs, but it is most unlikely that these will be correlated. At the best, if I may put it briefly, we may find correlation at the perimeter, whereas a year or two ago there seemed hope of correlation at the centre. But, for the present, it looks as if teachers will be faced with the same old multiplicity of requirements. The most obvious solution seems that " Matriculation " will grow a larger halo than ever to encompass the whole examination system.

As for specialisation! But I must not take up your time any more on general matters. Let us pass on to Mathematics.

As I said before, the Examining Bodies will have to decide very quickly what to include in the various syllabuses, and for time's sake the existing ones will have to be adapted where possible. At this stage, it is much more important that you who are practising teachers should say what you believe to be required than that you should listen too long to one who, in the nature of the case, has less of that particular kind of experience. At the Ordinary level I do not see much difficulty; I am not sure that the change is much more than a baptism service, performed with cold water! The present School Certificate syllabus seems fairly suitable, though I hope the retention of " Additional mathematics " will be allowed for the abler boys.

At Higher Certificate, or, as we should now say, " Advanced " levels, the matter seems less clear. In the first place, there is the loss of the present subsidiary mathematics, which is so useful to scientists and others who have not the time or, perhaps, the ability for a fuller course. The loss may be as severe to them as the loss of, say, subsidiary French or German may be to the mathematicians. For " Advanced " mathematics itself the present Higher Certificate papers will, presumably, suffice, though it is not immediately clear that separate syllabuses for scientists are envisaged. We must not forget, too, the complication that, no other subjects being compulsory, the weaker brethren may now be tempted to give excessive time to the critical subject and so cause an apparent rise in performance. Such a rise could react very unfairly, worse than at present, on those conscientious schools (and anyone who has much to do with education knows them to be many) who do try to give

their pupils a good general education. It would be a major tragedy if these schools were forced in self-defence to lower their standards against undue specialisation.

About " Scholarship " level mathematics I am less certain, and shall be glad to hear what you have to say. For my own part, I believe that the good boy needs to get his teeth into something beyond the present normal standard of " Higher Certificate " mathematics. University Scholarship examinations, for example, now include extra work (such as projective geometry), and some at least of the Higher Certificate Distinction papers follow suit. Let us not be too glib in saying that we can always pick the winner by setting specially appropriate questions on the normal syllabus : unless the paper is full of them, the candidates may answer the others! In a subject of steady development, like mathematics, a good boy should be continually enlarging his horizon, and not merely polishing up a number of set pieces till they shine with a light which, all too often, is merely a reflection. I hope the discussion this morning may help us to assess with some degree of accuracy just how far pupils may be expected to progress between the normal " Advanced " level and the " Scholarship " level.

I have probably spoken too much already, but I should like to conclude with two general points. First, I have not myself any experience of what the effect of the new proposals may be on the secondary modern schools. This is obviously of great importance, and we may note that the Report always uses the general term " Secondary Schools ", presumably to allow ultimately for both grammar and modern. The position of the modern schools has already been considered, and I would add just one remark : it is clear that any lowering of general standards will be of disadvantage all round, and the modern schools would not ask for it ; equally, I am sure that the secondary grammar schools will agree that the false status which has become attached to " School Certificate " (or its wicked Uncle " Matric ") ought to disappear, so that all types of schools can work together, in harmony if not in unison. Any sound examination system which encourages that harmony is to be welcomed, and there is certainly room for reform.

My second remark is to recall one of the greatest unsolved problems of modern examining—how to decide what part of a pupil's school work shall be subject to examination and what part he shall be left to enjoy in peace. If it were *certain* that in his last two years at school every boy would in fact be covering the ground at present devoted to the subsidiary subjects (and, indeed, more when possible) then the objection to their removal would be much less cogent, and many of the fears about premature specialisation would be removed. But we are in difficult waters, and it is perhaps wiser to change the metaphor and to pass the ball to you.

The President (**Professor G. B. Jeffery**) : I wonder whether it might guide the discussion if I spend a little time indicating what seem to me the relevant matters of urgency in this very large field. I am one of those responsible for the recent Report of the Secondary Schools Examinations Council. Therefore, it would not be right or proper for me to embark on a defence of the Report, but I think it should be remembered that the origin of it was in Circulars 103 and 113 of the Ministry of Education in which the then Minister, with the impetuosity which endeared her to some of us, said she would have no more nonsense ; that the gentle correlation by the Secondary Schools Examinations Council which had existed in the past was not enough ; and that there would be a spot of direction and control. But it had to be made clear from the University point of view that the Universities were in this examination business as a result of the cooperation they had developed with the teaching profession because they felt that they had a contribution to make to

education in that way ; and that if a situation ever developed to anywhere near the point at which the Universities or their examining bodies were mere agencies of the Ministry for the purpose of carrying out direction and control of an examination designed by the Ministry of Education, then the Universities would just stand down and the Ministry would have to do the job itself or find somebody else to do it. That had to be made clear, and I think it was.

The next point is that the Report of the Secondary Schools Examinations Council is advice to the Minister as to the extent to which he should go in that direction of control. It has been said—it is not for me to say—that on some points the advice contained in that Report is vague. In so far as it is vague, presumably the advice is that the direction and control should be vague. There are many points left completely unsettled in the Report of the Secondary Schools Examinations Council. Presumably, then, the advice to the Minister is that on those unsettled points there should be no direction and control; they are points which should be left to the examining bodies. So in considering the Report as a whole, you must weigh any omissions and any vagueness that you discover in that light, because they represent the additional liberty left to the examining bodies.

The next point I want to make is in regard to the position of compulsory mathematics ; this position is not made as clear in the Report as I myself would have wished. The Report says : make this Examination a subject examination from first to last, with no compulsory subjects and no minimum number of subjects. I firmly agreed with that recommendation and I might say that the whole of the Secondary Schools Examinations Council in making that recommendation had firmly at the back of their minds that the due balancing of the school curriculum of every child is a matter of the very first importance. Whether it should include mathematics or not include mathematics is not a trivial matter but one of the very first importance. But looking back over the experience of the group requirements of examining bodies, we came to the conclusion that this was not a problem that could be solved through the regulations of examining bodies. The responsibility should be placed fairly and squarely upon the teachers and the schools. The due balancing of the curriculum will not be prescribed by regulations, not because it is an unimportant problem but because it is the problem of the schools and not of the examining bodies.

There are two main lines of development here. What do you think, in your experience as teachers of mathematics, of this Report as far as it goes? Are there points of it with which you disagree ? Those are matters upon which you could properly make representations to the Ministry of Education as soon as you have got them into right order. Next, there is the very wide field you have been discussing this morning of the content of the syllabus. Those are matters which at the moment do not rest with the Ministry but will have to be taken up, without delay, by the examining bodies. I am sure I can say for the body of which I am Chairman that we would welcome most heartily representations from a body such as this as to the framing of new syllabuses to fit the new circumstances. There may well be a transitional period in which we take over the old syllabuses, more or less, but it is clear that we should at once set to work to bring those syllabuses more fully into line with the needs of the schools at all levels. That matter is not quite so urgent, but it is urgent and it is a matter for the University examining bodies and not, at this stage, for the Ministry of Education.

Finally, there is the question of what the University entrance requirements are going to be. Some of us have been much concerned to get the Universities together, not with the hope that we are going to get a complete flat scheme applicable over the whole country, but in the belief that we can get a good deal

closer than we are at present to something like a common scheme of entrance requirements for the Universities. I am not so hopeless about that as Dr. Maxwell appears to be, because later in this month in this Institute of Education we are going to have a conference of the bodies represented to see how far we can get in that direction.

I have said this to indicate something of the relative order of urgency of some of our problems.

Mr. W. F. Bushell (Birkenhead): The President has indicated one subject at least to which I might add something, and may I, first, say a few words in regard to the Report itself, with special reference, or almost entire reference, to the grammar schools; I have no right to speak of the problems of girls' schools or those of modern secondary schools. It is difficult in making these remarks to separate mathematics from other subjects, because the general remarks one makes in regard to mathematics naturally and inevitably concern other subjects also. I definitely regret to find that to-day, for some reason, these external examinations appear to have got somewhat into disfavour. I speak from the standpoint of one who started teaching some forty or even more years ago. Probably there are some in this room who started before, or even considerably before that. But looking back forty years when we did not take these certificates—of course, some schools did, but many did not—and had purely internal examinations at the end of the summer term, and comparing that with to-day when all schools are having these examinations—school certificate, first-year higher certificate, second-year higher certificate—there is a very striking difference. The supreme advantage of this modern development of exams lies, it seems to me, in the tonic effect—and that is a word I would like to use and emphasise—not only upon the boys but upon the teachers. It really has been an immense tonic effect over these last thirty years—and I say "thirty" because I think it was in 1917 that the school certificate actually started with its credit system in the way we know it to-day. Many of these alterations, as I have pointed out previously, do start at the end of great wars when we feel we have to do something for the sake of future generations. They have got somewhat into disfavour. The theory is that we are shackled. I believe Mr. Langford used the phrase "the so-called tyranny of examinations". I was interested to hear him say "so-called", showing that he did not necessarily associate them with tyranny. I have sometimes asked colleagues who have spoken of being shackled precisely how, in their mathematical syllabuses, they would alter the course if they were not shackled, and could have whatever syllabus they liked and a purely internal examination. The answer has been very meagre. Although I may desire some slight alterations in the examination for which I want to prepare boys, yet those alterations would be of a very slight character and I venture to suggest that the syllabuses brought before us by the various examining bodies have been, on the whole, of a very high character, and indeed often superior to those that internally we could ourselves devise.

Consider what has happened in these thirty or forty years. Think of the number of boys who learnt trigonometry or calculus some forty years ago compared to the immense numbers learning these subjects now. Ought we to deprecate the effect of these examinations in the secondary grammar schools, realising the immense advance in the scope of mathematics for which they have been largely responsible? Under this new circular which has just been issued, of which Dr. Maxwell has spoken, it would seem that there is only to be one year with some external examination. Under Section 32 a boy may take his ordinary advanced, or scholarship papers, which would imply that he should not take it more than once. Dr. Maxwell used what I thought a very good word: he said that the ordinary examination which a boy should take at

the age of 16 or 15½ supplied a very useful *barometer* as to how the boy was getting on. If the boy is to wait until he takes the scholarship paper, or something of that sort, at a substantially later age, that barometer will not exist, which is, as I see it, a misfortune. We are, however, told that internal examinations are going to settle all these difficulties. I wonder what colleagues in this room feel about internal examinations? I am bound to say I have always had my suspicions of them. Internal examinations do not possess the prestige given to an external examination; they are often carelessly done and prepared. I wonder if we are going to get a new standard in internal examinations, but I should regret the partial abolition of our external examinations even if we put in their place internal examinations, however good.

There is one more section in the Report which has not so far been quoted, Section 30, which says that the schools alone are to be in a position to decide what the boys ought to do and ought not to do. That section gives the schools complete responsibility and control over the boys and what they are to do, and yet another section forbids them to take these examinations until they have reached a certain age. Apparently they have got to be 16 on September 1st of the year in which the examination is taken. I believe that is the month and the date. That is to say, if they were taking the examination a month or two earlier they would be just under 16. I do not say I necessarily want a boy to take it before that age, but how far does the school retain its freedom when they put down that date with the strictness which I have just quoted?

And then, apparently, there is to be another sort of school certificate, a kind of internal school certificate of which we have heard before, in the form of school records. There is a good deal of talk about school records: they have got to be very full; they have got to be open to everybody to inspect. Am I right in calling it a form of internal school certificate? I should very much regret having to write so elaborate a school record as is apparently wanted, and to discuss a boy's mathematical ability for the employer, for the parent, for the boy and for everybody else to read. Of course, it is extremely difficult for any one of us to give the exact truth. Those of us, and there are some, who have had to write hundreds of open testimonials for boys who are leaving school and applying to employers, know the extreme difficulty and have little belief in the open testimonial. So I fear many of us would have little belief in a school record which will probably tend to represent the boys, who are not worth much, better than they are.

In one of the papers read it was said that the Mathematical Association should encourage a modern secondary membership of the Association. I do want to associate myself very strongly with that. This Asociation has played a very honourable part for a great number of years in the development of mathematical teaching. If we do develop, perhaps because of this Report or for other reasons, a modern secondary school membership, who will add their knowledge, advice and sympathy to our counsels, an important and valuable result will have been accomplished. Mr. President, I wish, particularly, to express my great regret that this very fine system of examinations which has been developed, which has done, which is doing, and could do so much for the grammar school education of this country should, apparently, be suspect for reasons with which I certainly am not in agreement.

Mr. C. T. Daltry (Institute of Education): I wish to draw attention to a point which does not seem clear. I would like to read par. 32 of the Report: "It would be contrary to the intentions of the system for any pupil to sit successively for ordinary advanced and scholarship papers in the same subject." The intention of this is that a pupil should take a single examination as late in the school career as possible. I do not think that members fully

appreciate that. It seems that many are still thinking that we have the three levels; and that we have to do something at each level, and that each pupil has to pass through three successive hurdles. That is not the intention. I felt it might help if that were made clear before the discussion proceeded further.

The President: While the Report does not forbid a candidate in three successive years to take a subject at Ordinary level in one year, at Advanced level in the next year, and at Scholarship level in the third year, it does say that this is contrary to the intentions of the Report.

Mr. K. S. Snell (Harrow): That last point is one on which I think we should put forward a special recommendation. I notice it is possible to take the ordinary papers in some subjects, particularly those in which the boy is not specialising, and postpone to a much later date taking the advanced papers in the subject in which he is specialising. My feeling, and possibly the feeling of a large part of this audience, is that it should be made more permissible, encouraged more, for a boy to take the ordinary papers at, approximately, the age of 16, and some advanced papers two years later.

As regards the general question of the examination, the only other point I had in mind was that of the age-limit which has caused so much criticism amongst teachers. It is clearly put there to prevent this examination being taken by the majority of modern school pupils and I think we would be all in agreement with that intention. Therefore I think we should be wasting our time in trying to get that age-limit altered. I gather it is one thing on which the Ministry of Education was very firm, and if we are in agreement with their intention I think we should make arrangements to agree with the age.

As regards the details of the examination from a mathematical point of view, I think we had put before us most excellent suggestions by Mr. Langford and Miss Pendry. In the basic mathematics for all, to be taken either in the first, second or third years, I felt that the suggestions put forward by Mr. Langford took the cream off mathematics to a deeper extent than we should encourage or agree with. I consider the syllabus was, in fact, definitely too wide. Consider the proposed syllabus in Geometry. Speaking as one who starts teaching boys at the age of 13½, I should be completely astonished if boys arrived at that stage after two to three years of this course knowing as much as Mr. Langford suggested. I suggest cutting out everything to do with the circle, inasmuch as it involves geometry, and mensuration, and so reduce what he put forward. As regards the second stage, dealing with the fourth and fifth years, I most strongly agree with what Mr. Langford said, that the additional mathematics should be kept as well as the ordinary mathematics. I would not agree that mechanics should necessarily be a third alternative, but I do feel strongly that it should come into the additional papers.

As to the syllabus for the ordinary papers, we have had in the Jeffery Report excellent suggestions, which are now in process of experiment in the various alternative syllabuses. This report has been produced by examining bodies, and it is on that we should base our syllabuses. This involves, firstly, an insistence that papers should be general mathematics papers and not in individual subjects. Secondly, as regards the content of that syllabus I have always felt a little doubtful about calculus, and I have always wanted to put forward the possibilities that the beginnings of mechanics in the form of simple statics and simple kinematics, as an illustration of school mathematics, should come into that syllabus as a possible alternative. I think we need to develop that syllabus on lines which broaden the possibilities but which certainly do not deepen—that is to say do not go any farther in advance in any one subject.

As to the last point raised by Dr. Maxwell with regard to work to be done before scholarship standard, I do not wish that to be limited. I agree with Mr. Langford that some projective geometry is essential and I think, as

scholarship papers must always involve many alternative questions, that if the present papers set by Universities allowed a few more questions of an easy type, we would not need to suggest reducing the actual syllabus which they cover.

Mr. L. R. B. Elton (Holloway School) : I am still a little puzzled by par. 2. If the pupil is to take only one examination before he leaves school, it means he has to carry all the subjects in which he wants to be examined up to that age ; otherwise he will never be examined on some subjects at all, although he may like to be able to show later that he had taken them and might have passed. If, on the other hand, he takes some of the subjects which he wants to drop at an earlier age and only those which he wants to go on with at advanced level at a later age, it would be difficult to persuade the boy that he should only do the subject which he is not good at, or not so good at, at 16 and leave the subject at which he is really good until 18. He will want to show up in the subject which he can do. He will not want a certificate at 16 with his best subjects missing.

As regards the syllabus, I feel too little has been said about the very good and the very bad at the grammar school. I do not know whether I have had a particularly unfortunate experience, but I have for the past term been teaching in a grammar school a class of average age 15 who find it very difficult to produce a correct answer to an addition sum. Probably this is not a singular experience ; boys like that do appear in our grammar schools who will not get beyond Stage A by the time they leave school. I feel provision must either be made in the grammar schools for them or else the standard of grammar school entry must be changed in such a way that such boys do not get into the grammar school.

As regards the particularly able pupil, I quote the example of one boy who, being at a public school and, therefore, freer to take examinations at what age he likes, passed last year the school certificate with eight distinctions and one credit at the age of 15·0. I do not think that can be classified as undue specialisation or cramming, because the boy did not work very hard. There just happen to be boys like that, and it seems hard to make them all waste time for at least another year until they go on to something more interesting. That particular boy went on to produce a very good higher certificate in one further year. Boys such as he can do it at the public schools, but not at the grammar schools on the whole, because of staffing difficulties and so on. That places grammar schools at a very great disadvantage to public schools when it comes to open scholarship examinations at Universities. The public school boy who wanted to sit for an open scholarship at Cambridge took his school certificate at 14 and passed ; the higher certificate before 17, and the second higher certificate at 18 and then the scholarship. At the first higher certificate I do not think any boy knows sufficient mathematics to make a good show at a Cambridge open scholarship in mathematics. As long as Cambridge open scholarships are held in December, two higher certificates or the equivalent number of years spent in the VIth form seem to be necessary, with consequently earlier specialisation at the age of 15 for very good pupils.

Dr. O. R. Baldwin (Bristol) : Mr. Snell has accepted the age qualification. We had this discussion in the Bristol Conference when Sir Philip Morris, who is another signatory of the Report, agreed with the head-mistresses present that after the statement of the age there should be another clause saying : " Or has completed five years in a grammar school." Speaking as the head of a mixed school, I pointed out that children in the Bristol area are admitted at the age of $10\frac{1}{2}$ and after five years (at the age of $15\frac{1}{2}$) many of them would leave school. There is no way of compelling them to remain or pretending to compel them, and therefore there would be a class in which half could take the

examination, but the other half would leave, being unable to take the examination because of this age qualification. We think that if there is to be an age qualification at the one end there should be a change of age when the pupils enter; that you cannot stipulate a definite age at the close of the school career for most pupils unless you also state a corresponding age upon entry. I should like the head-mistresses throughout the country to follow the head-mistresses of the Bristol area in getting the signatories to the Report to agree that they made an omission there; that five years in the grammar school is the qualification and not the age which they have put down. In my opinion the insistence on age has been made by University authorities who are concerned with matriculation.

Mr. F. W. Kellaway (Apsley): I am sorry to find this discussion, interesting as it is, concentrating so much on the grammar schools. I cannot supply any answer, but I hope there are those here from the modern schools who will take up the point as to whether it is right to debar any student at a modern school from taking these examinations; whether you do want an examination for them or not. Of course, though I am not an advocate of a tripartite system, we have the grammar school, the technical school and the modern school and so far the secondary technical school has not been mentioned.

Mr. G. L. Parsons (Merchant Taylors' School): I call attention to one point in regard to the age problem which makes me very diffident about accepting Mr. Snell's hypothesis that we should agree with the age-limit proposition. The age-limit proposition in the Report stands in two forms: first, in the form that the age must be 16 on September 1st of the year in which the examination is taken. That statement itself exists in two forms, one in 21 (*g*), in which it says that no grant will be paid for a pupil under that age, and the other in the recommendations on the back page where the prohibition is more absolute. But the second point relating to the age-limit to which I particularly wish to draw attention is that the Report further goes on to state that it is the hope of the Minister of Education that this age-limit will be raised. If the age-limit is raised, it seems to me that the argument in favour of agreement with it becomes very much weaker. The situation that has been mentioned by one speaker already is certain to arise, that the pupil will want to take some of his subjects at the " ordinary " level and get them out of the way, so to speak. That may be possible if it is to happen at the age of 16, but if it is not going to be possible before the age of 17 it means that the pupil will have to devote a fair proportion of time to a number of subjects, in which he is not going to follow an advanced course, for longer than it is fair he should do so. Thus he will deprive himself of valuable time for those subjects in which he is more specially interested.

Mr. K. R. Imeson (Rochester): The effect of this age-limit does mean that some boys will be prevented from taking any subjects at the ordinary level until about the age of 16½. But supposing a boy has studied French for five years and wants to go on to the science side of the VIth Form, under present conditions he would take the school certificate in French and then specialise in the VIth Form in his scientific subjects. He would probably keep up his French but with a different method of approach. He would cease studying the dry bones of the subject, which I think is what he has studied up to school certificate, and branch out in another direction. He might possibly even take the subsidiary French in the Higher Certificate, and I do regret, as Dr. Maxwell does, the abolition of this subsidiary standard which does not correspond in any way to the School Certificate credit standard and yet serves at present a very useful purpose.

To take another subject, history or geography. A boy who is going to study mathematics in the VIth Form takes history and geography in the School

Certificate and then branches out into a new direction and takes, possibly, a course in current affairs in the VIth Form. Under the new system he will be prevented from doing that for another year at least, so that the general education of the science specialist will suffer severely.

Mr. P. Vermes (Farnborough Grammar School and Birkbeck College): I have been so lucky as to be admitted to the English teaching profession. That was about seven years ago prior to which I had some knowledge of education on the Continent. I find this Report is an approach to the system in use on the Continent, but I regret to state that it takes over almost all the drawbacks to be found in the continental system, and it dispenses with almost all the advantages of the English system. I greatly admire many things in English education. For instance, I would very much regret the abolition of the external examinations which are fair both to the student and to the teacher. They allow the possibility of checking the results of the teacher's work, and they provide a stimulus for him in that they help to keep his standard of teaching up to date. On the Continent there was a very rigid system; all subjects which should be taught in a course were strictly prescribed: failure in any subject meant that the student failed to go on. As a result, his general education was much improved. Under the present project the pupil can make a choice according to the subjects in which he is interested and may obtain a certificate in them without having the benefit of a general education.

Mr. E. J. Atkinson (Newport): Dr. Baldwin at Bristol, taking boys at $10\frac{1}{2}$, runs a "five years" course for his boys to take the examination at $15\frac{1}{2}$.

I left the London area, where we had a four year course, and took over a school in South Wales which was then, and still is, a depressed area. Here, too, our pupils enter at 11, and for most the "four year" course is the usual thing, and these boys can and do take the examination at the same age as Dr. Baldwin's. Why stipulate 4 or 5 years?

In my own case I have 100 entrants a year. Some leave before the end of 4 years. Increase the age to 16 and I should not have anything like the number remaining to take the examination, and I fear my colleague in the girls' school would have even less. We have to fight hard now to persuade the pupils to remain long enough to take the School Certificate, very particularly so in the case of a boy who requires a fifth year.

Firstly, if examinations are to go, I do not wish to be a headmaster who has to certify that this or that boy is of a certain standard. I do not wish to be subject to the pressure I might get from persons of influence and others attempting to get good reports.

Secondly, if we consider the examining bodies and the examination syllabuses, are not the examiners themselves men with teaching experience in school or university and do not both the subject and teacher associations play a full part in drawing up the syllabuses; thus what complaint have we?

I know much has been said with regard to the record card. In another place yesterday, the record card, like this age qualification, was strongly criticised.

I hope when this Report is under full discussion we shall not cast aside a form of standard certificate which now has very wide acceptance, and throw it back upon the poor headmaster to give his own certificate.

I feel it is our job to fit our pupils that they may take their places qualified to enter training for any profession they wish to follow. If we allow them to go out from our grammar schools without this qualification, which is the same as that a boy at Harrow can get, we are doing untold harm to the poorer boys. I think that the examinations to-day are fair and reasonably satisfactory.

I hope that the boys for whom I am now responsible may always have an equal chance with the boys from the public or any other schools.

The President : We are getting within sight of the time when we must close the meeting. I suggest that we leave questions of syllabuses and their content in the hope that the Council and the Teaching Committee will at once take up these questions with a view to making representations to the examination bodies. I am sure that would be useful, and that it would be the intention of the Council to proceed in that way. The Council wondered whether out of this morning's discussion there might arise a general opinion on certain specific points on which you would wish representations to be made on your behalf. I have heard during the discussion three points which might emerge in that way, but it is for you to say what your reaction is. The first two points are closely linked together : the relation of the whole of this business to the secondary modern school. Is the examination in any form to be applicable on any scale to modern schools? In the terms of the Report it is completely open, but I think it is also an open secret that a good deal of the controversy surrounding the Report was due to representations from certain quarters that at all costs secondary modern schools should be saved from examinations and one powerful reason for the age-limit was to cut out the modern schools. The present leaving-age is 15, and if you design an examination for a course lasting up to 16, it is obviously not desirable that schools should try to rush that examination in before the age of 15. If and when the school-leaving age is raised to 16, then on the Report as it stands it would be quite right and proper for secondary modern school pupils to take certain subjects in the examination at ordinary level. That is wrapped up with the age-limit. Is there here any opinion on the relation of this examination to modern school work? Is there any opinion on the age-limit, either as it stands at 15 years and 8 months at the date of the examination, or on the present intention of the Minister of Education that that age of 15 years and 8 months is an interim stage towards a 17-year age-limit.

The third point I noticed during the discussion was a certain expression of regret at the abolition of the old subsidiary level and the feeling that the ordinary level taken in the course of the VIth Form is not an adequate and proper substitute for the old subsidiary level.

If there is a general opinion on one or more of these three specific points, I think we should take them in turn, concentrate on them and try to formulate some kind of opinion that we can follow up.

After some further discussion, the following resolutions were passed, with about 250 members voting.

1. " This Association is of the opinion that any control exercised in respect of the age at which candidates will be entered for the examination is contrary to the spirit of paragraph 30, and recommends strongly the abolition of any reference to an age-limit." (Passed with 5 dissentients.)

2. " This Association deplores the proposal to abolish the subsidiary level of the old Higher Certificate examination and considers that none of the proposed papers would take its place." (Passed with 1 dissentient.)

3. " This Association hopes that the interpretation placed upon paragraph 32 will not preclude the possibility of a pupil taking Mathematics at the Ordinary level and taking the same subject at a higher level in a later year." (Passed *nem. con.*)

Mr. Riley : The question of modern schools has been mentioned so often this morning that I should rather consider myself in the way of replying to the discussion on the paper I read on behalf of Mr. Brown. We have been dis-

cussing the Report of a Council considering examinations in secondary schools. I was glad that one or two speakers underlined the fact that that applied to secondary schools and not merely to grammar schools. The paper by Mr. Brown dealt with modern schools, and I think only one speaker mentioned the word "technical". I am glad it was mentioned, but it is too late now to embark on any discussion on these points. Some of my friends might have expected me to rise on the motion in regard to the age-limit. I did not do so because it is my experience that any regulation ever made can be got round, and I know of at least one secondary modern school which is already proceeding to get round that regulation as it stands with the age-limit at 16, because there is nothing to prevent a child staying on now without the leaving-age being raised and taking the examination. The safeguard for secondary modern schools—and a safeguard is very necessary—is in Section 30 which was first quoted by Mr. Bushell and subsequently by several speakers, which says that schools alone are in a position to decide which is best for their pupils; they need the utmost flexibility to give effect to their judgments. The secondary modern schools need that freedom. We hope they will value it.

Mr. Langford referred to the necessity of persuading parents from forcing their children to take examinations for which they are not suited. I take it he was referring to grammar schools. The danger is in secondary modern schools, that parents seeking the kudos which they feel does not attach to the secondary modern school attempt to gain it by forcing their child to take the examination which has been designed largely for grammar schools.

I liked the phrase of Dr. Maxwell in which he said we need continual expansion of experience rather than the polishing up of a few set pieces. You can polish up a few set pieces in any type of school at any time. We do not want that to happen in the secondary modern school; we want complete freedom, and I think we shall do something in the direction of framing something in the nature of an internal examination externally assessed. But these things take time, and under the existing Report I see no danger ahead for the secondary modern schools, and I do not think the resolution you have passed about the age-limit will affect them one way or the other.

Mr. Bowen (Kingston-on-Thames): I meant to interpose in the discussion when Mr. Kellaway mentioned the fact that the discussion had been almost entirely confined to grammar schools; that some mention had been made of secondary modern schools and no mention of secondary technical schools. I would like to add an authoritative word from the point of view of technical schools, speaking as headmaster of a technical school. Last year at the Conference of Headmasters of Technical Schools in Birmingham, it was decided that no general technical schools would have anything to do with external examinations. I am not here going to debate the merits or demerits of external examinations. It was agreed that technical schools which could at the present time—and some do—take the present school certificate should develop their own internal examination externally assessed rather after the style of the present national certificate in engineering. It was almost unanimously agreed by that Conference that the aim of the technical schools, which will be recruited at 13 or 14 plus, would be to avoid imitating the grammar schools in any way by taking this school certificate or its modified form, and that we should aim at setting our own examinations in the schools, which would be externally assessed by some responsible body and which would be nationally recognised.

The President: That gives me the clue to what I should say as my last word in the office of President. Our discussion has been focussed mainly on the reaction of these examinations on grammar schools. One thing which stands out from the Report of the Council, which you received at the first session of

this Annual Meeting, is the growing interest of the Mathematical Association in the teaching of mathematics not only in grammar schools but in secondary modern schools and in technical schools. Whatever this Association may have to say in relation to examinations, which constitute only one small part of the business, it is interested in the teaching of mathematics over the whole field. I would like to bring the Universities in because there is great need for that. The interest of the Mathematical Association in the teaching of mathematics at all levels and in all kinds of schools and colleges is growing, and I am sure will grow. And so, although our friends from modern and technical schools may feel a little neglected in this morning's discussion, they are by no means neglected in the general work of the Association ; neither are they neglected in the interest of us all. On that, perhaps, we can close our meeting. The Council has appointed a Committee in advance which will take the resolutions you have passed this morning and incorporate them in an appropriate representation to be sent to the Minister of Education.*

* As reported in the February *Gazette*, this has been done.

NOTICES.

MATHEMATICAL TABLES FOR SCIENCE AND INDUSTRY.

It is considered that important investigations in the physical, chemical and engineering sciences, in mathematics and in industry may be held up by the absence of relevant mathematical tables. Several important tables of a fundamental nature have in the past been produced by the British Association through its Mathematical Tables Committee. The work in this field has now been transferred to the Royal Society on the invitation of the British Association, and a special Royal Society Mathematical Tables Committee has been established to continue and extend the earlier work. This Committee would be glad to receive suggestions from industrial research groups, from individual investigators and from Service departments relating to existing needs in their special fields. Requests for assistance may be met either by advice as to the most economical way of producing the desired tables, or by an undertaking to produce the tables under the auspices of the Royal Society, or, in exceptional cases, by a grant towards the cost of tabulation or to make possible the publication of important tables which may exist only in manuscript. In the first instance inquiries should be addressed to the Assistant Secretary, The Royal Society, Burlington House, London, W. 1.
July 1948.

TEACHING OF STATISTICS IN SCHOOLS.

The Committee of the Royal Statistical Society on the Teaching of Statistics has under review the question of teaching the subject in schools. It is believed that much useful information could be obtained from schools where there has been some experience of teaching Statistics in one form or another. Any information from members of the Mathematical Association as to the names of such schools or of individual masters would be welcomed by the Committee ; it should be sent to the Assistant Secretary, Royal Statistical Society, 4 Portugal Street, London, W.C. 2.

REVIEWS.

Theory of Functions. By J. F. RITT. Revised edition. Pp. x, 181. 16s. 1947. (King's Crown Press, U.S.A.; Geoffrey Cumberlege, London)

Ritt's elegant construction of the real number system on a decimal basis effects so great an economy in time and exposition that the universally accepted approach to the theory of functions through the Dedekind section will have great difficulty in maintaining its established position. The key theorem in this line of development is the existence of the least upper bound of a roughly bound set of infinite decimals, which is proved by the successive selection of maximum digits, establishing a technical equivalent of Dedekind's theorem with a minimum of abstraction.

The method of successive selection of maximum digits was first employed by Dienes in his *Taylor-Series*, but there the application was confined to monotonic sequences. Ritt's wider use of the technique simplifies the arithmetical operations with decimals though it results in some loss of what might be called comparative finitism. Both Ritt's treatment and Dienes' share with the Dedekind theory the disadvantage of making the limit of a convergent sequence an indirect construction rather than something which is generated by the sequence itself.

The natural numbers are accepted as having intuitive meaning; decimals are introduced by setting up a formal calculus of operations with number patterns. No attempt is made to bridge the gap between the natural numbers and these decimal patterns despite the obvious link afforded by the sequence concept which is accepted as primitive. The familiar passage from natural numbers to terminating decimal followed by the definition of $a_0 \cdot a_1a_2a_3 \ldots$ as an abbreviation for the sequence $a_0, a_0 \cdot a_1, a_0 \cdot a_1a_2, a_0 \cdot a_1a_2a_3, \ldots$ would seem to be preferable. Moreover the insistence upon the separateness of the complex number $(a, 0)$ and the real number a, later in the book, is totally at variance with the formalised thesis in the first chapter.

The subject of the book is the elements of the classical theory of functions of real and complex variables, with an emphasis on fundamental concepts and processes that is very rare in an elementary text. There is much in the presentation that is fresh and novel, including a new proof (amongst several others) of the fundamental theorem of algebra and a most attractive derivation of the maximum modulus principle from Rouché's theorem. The analytic character of the integral $\int_c \{F(w)/(w-z)\}dw$ is stressed and finds a number of interesting applications, taking the part of Morera's theorem, for instance, in the proof of the differentiability of the limit of a uniformly convergent sequence of analytic functions. The treatment of the corresponding problem in the real variable (pp. 45-46) unfortunately falls far below this standard of elegance.

There is a most important and valuable analysis of the variation of the amplitude of a continuous function along a continuous curve (pp. 171-173), a matter which has not previously received the careful attention it deserves.

The rather surprisingly elaborate proof of Cauchy's theorem,

$$\int_c dz/(z-a) = 2\pi i,$$

on pp. 105-107, is an unfortunate result of the delay in introducing the circular functions and the total exclusion of the elementary integral formula

$$\int_{z=z(t)} f(x)\,dz = \int f(z(t))\,z'(t)\,dt.$$

Another consequence of this delay is the serious mistake on p. 104, § 4, where the *amplitude* of a certain complex number is said to be 90°.

The definition of a maximum value of a function (p. 27) raises a point of nomenclature. Is the condition for a maximum value at x_0 : $f(x_0) \geqslant f(x)$ for all x in the neighbourhood of x_0 ; or should it rather be : $f(x_0) > f(x)$ for all x near but different from x_0? That is to say, should the term maximum be confined to a value which exceeds all near it, or is a maximum merely a value that is not exceeded in its neighbourhood?

On p. 41 the definition of *rearrangement* is inadequate, for to define

$$u_{n_1} + u_{n_2} + \dots$$

to be a rearrangement of $u_1 + u_2 + \dots$ if $n_1, n_2, \dots$ is a rearrangement of $1, 2, 3, \dots$ is just to define a concept by one of its instances. Two further small points perhaps worth noting are the omission of the definition of *equality* for complex numbers on p. 55 and the loss of the conditions for equality in the crude proof of the triangle inequality (p. 57).

The book is a very successful example of lithographic production, easy to read and handle, with large pages and a spiral spine. It is type of production not very suitable for general library purposes, but it makes a valuable and reliable introductory course in analysis available to students at a comparatively modest cost.

Errata.

p. 18, l. 21 : for $b, c)$ read (b, c).
p. 19, l. 10 : for a_1 read a_i.
p. 23, § 14, l. 2 : for $A = B$ read $A \neq B$.
p. 32, l. 3 : for $a = a$ read $x = a$.
p. 39, l. 10 from end : for $+a_1$ read $= a_1$.
p. 58, l. 12 from end : for y read y_n.
p. 65, § 1, l. 10 : for $|h| < \delta$ read $0 < |h| < \delta$.
p. 70, l. 2 : after *in particular* add *if* $z \neq 0$; l. 14 : for $z_0 - z_0$ read $z_1 - z_0$.
p. 142, § 6, l. 1 : after $f(z+h) = f(z)$ add *for all* z.
p. 156, § 4, l. 5 : for *linit* read *limit*.
p. 165, § 5, l. 2 : for $f(z)/f(z)$ read $f'(z)/f(z)$.

R. L. GOODSTEIN.

Integration in Finite Terms; Liouville's Theory of Elementary Methods. By J. F. RITT. Pp. vii, 100. 15s. 1948. (Columbia University Press ; Geoffrey Cumberlege, London)

In his Cambridge Tract, *The Integration of Functions of a Single Variable*, probably the only other book in English dealing with the subject of the work under review, Hardy wrote : "It was Liouville who first gave rigid proofs of whole series of theorems of the most fundamental importance in analysis—that the exponential function is not algebraical, that the logarithmic function cannot be expressed by means of algebraical and exponential functions, and that the standard elliptic integrals cannot be expressed by algebraic, exponential and logarithmic functions. That such theorems require proof is too often altogether forgotten." Professor Ritt has given a clear and modern account of the work of Liouville and of later work, including his own, on these and related subjects.

Chapter I contains a discussion of the elementary functions of one variable : that is to say, the functions generated by rational processes, by solution of algebraic equations, and by exponentiation and taking logarithms. A hierarchy of orders of such functions is considered : algebraic functions of sets of functions in any order belong to the same order, but exponential or logarithmic functions belong to the next higher order. These functions are

considered in the complex plane, and their definition discussed with great thoroughness. Chapter II uses these definitions to discuss the conditions that an algebraic function may have an elementary integral: the main theorem is Liouville's, that if $y(x)$ is an algebraic function with an elementary integral, then that integral is of the form

$$\int y(x)\,dx = v_0(x) + c_1 \log v_1(x) + \ldots + c_r \log v_r(x), \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where the v's are algebraic functions and the c's are constants. Abel's theorem, that the v's in (1) are rational functions of y and x, is then proved.

By means of the theory of residues of algebraic functions on their Riemann surfaces, very simple and elegant proofs that the elliptic integrals of the first and second kind and Chebyshev's integral are non-elementary are given. (It is worth mentioning that some of the theorems cited by Hardy become almost trivial when these complex variable methods are used.) Chapter III gives a generalisation of these theorems, due to Ostrowski, to differential fields of functions—that is to say, to sets of functions which are fields in the algebraic sense and which are closed under differentiation. This is applied to establish Liouville's theory of the integrals of exponential functions, and to prove, for example, that e^x/x has a non-elementary integral. Chapter IV treats miscellaneous questions in the theory of elementary functions—including the proof of the existence of elementary functions of all orders, and of the non-elementary nature of the function y satisfying Kepler's equation $y - a \sin y = x$. After preliminary matter in Chapter V, Chapter VI discusses the integration of differential equations by quadratures. Here a new hierarchy of orders of functions is introduced, in which integration is added to the operations by which functions of one order may be produced from the order preceding it. It is proved that the equations of Riccati and Bessel, for example, cannot be solved by a finite number of quadratures save in the classical cases. Chapter VII deals with solutions in implicit form in terms of elementary functions, and with both implicit and explicit solutions of algebraic differential equations. Chapter VIII gives further results on implicit equations, due to the author, including the interesting result that if an integral, w, of an elementary function satisfies an elementary equation $F(w, x) = 0$, then w is itself elementary: this is used to discuss implicit solutions of second order differential equations.

For the most part the book uses only the analysis of an honours course: the principal tools are the Weierstrassian definition of an analytic function, and the expansions of a function on its Riemann surface. The ideas used are explained carefully and thoroughly, and the book is, in general, very readable. It has been produced by photography from typescript, but the reproduction is almost as clear as print. Anyone interested in analysis will welcome Professor Ritt's account of this subject.

In conclusion, I may mention a few misprints: On p. 14, line 8 from bottom, "$\mu = T - 1$" should be "$T = t - 1$"; p. 25, line 7 from bottom, "$u(x)$" should be "$u(\theta, x)$"; p. 26, line 4, "c_i" should be "c_j", and in eqn. 28 "$u(x)$" should be "$u(\theta, x)$".

J. L. B. COOPER.

Elementary Differential Equations. By L. M. KELLS. 3rd Edition. Pp. xiv, 312. $3. 1947. (McGraw-Hill)

This textbook by the Professor of Mathematics at the U.S. Naval Academy is intended for engineering students who have some grounding in the calculus. The elementary methods of solution of ordinary differential equations and of partial differential equations of first degree and some partial differential equations of second degree are treated. Some discussion of existence theorems,

without proofs, has been added to the book since the first edition, but the book is devoted mainly to methods for obtaining solutions. Space is devoted to explaining the derivation of elementary differential equations of mechanics, electric circuit theory and of heat conduction, and a large number of numerical examples in these fields are given. The treatment of mechanics employs the "slug" unit of mass, abhorrent to the theorist but now widely used by aeronautical and other engineers.

A short account of solutions of second order equations in series and of first order equations by methods of successive approximation, including the Runge-Kutta method, is given. The book would have gained by a wider use of these methods, which are more widely useful to engineers than methods of getting solutions in a closed form. J. L. B. COOPER.

Unified Calculus. By EDWARD S. SMITH, MEYER SALKOVER and HOWARD K. JUSTICE. Pp. x, 507. 21s. 1947. (John Wiley, New York; Chapman & Hall)

The authors describe this book as a first course in calculus which "for the most part presupposes a knowledge of the mathematics generally taught in the freshman year". One suspects that it is written with the needs of the engineering student in mind. As the title suggests, the presentations of differential and integral calculus are interwoven; the process is facilitated by the introduction of differentials at an early stage and by the free use of examples drawn from applications to mechanics and physics.

The topics dealt with range from the introductory ideas of functions and limiting processes, through the standard differentiation and integration properties of functions of a single variable, by way of approximate integration, indeterminate forms and infinite series, to partial differentiation, multiple integrals and differential equations. The treatment throughout is elementary, and the emphasis is on the introduction of ideas rather than the development of a polished technique. There is, for example, no detailed discussion of the harder types of standard integrals, no mention of the change of order of integration in a repeated integral, and no discussion of serious problems of change of variable in either integration or differentiation. Incidentally, the proof of the elementary result on the differentiation of a function of a function makes the familiar mistake of ignoring the possibility of a zero derivative.

The shortness of the chapter on approximate integration is disappointing in a book of this kind. Simpson's rule is the only method discussed and there is no indication of the magnitude of the error. The chapter on indeterminate forms, on the other hand, is perhaps over-emphasised. It is a pity to elaborate the relation, obtained from Cauchy's mean-value theorem, between the limit of a quotient and the limit of the quotient of derivatives, while ignoring the possibility of the evaluation of the limit from first principles as the quotient of the derivatives themselves. All that is needed in an elementary treatment could have been obtained from a simple application of the Taylor's formula with remainder that is given two chapters later. One would like to see greater emphasis on the comparison between the behaviour of the polynomial and exponential functions for large values of the variable. As it is, a limit such as $\lim_{x\to\infty} x^3e^{-5x}$ appears rather casually among the examples, to be evaluated by three applications of L'Hospital's rule.

The limitations imposed on a calculus textbook necessarily mean that certain fundamental results must either be stated without proof or used without explicit formulation. The authors of this book make suitable comments to call attention to their use of a number of such results, but they let some points go unremarked that could profitably have been similarly treated. There

is a rather uneasy presentation of the elementary properties of limits, no explicit recognition of the necessity of establishing the properties of continuous functions and no comment on the use of intuitive ideas of area and volume. In general, however, the exposition is careful, clear and concise; the material is arranged so that the argument develops steadily with increasing demands on the reader's experience; and throughout the text there are worked examples illustrated by numerous diagrams as well as collections of problems for solution. A table of standard integrals, a reference list of formulae from algebra, trigonometry and geometry, and tables of logarithms, trigonometric functions and exponential functions are provided at the end of the book.

An English student of natural science or engineering who works carefully through this book should be in a position to tackle with confidence most of the applications of calculus that he meets with in the last years at school and the first year at the university. The book is less well suited to the needs of the serious mathematical specialist, however, for, not only does it provide less detailed experience than he needs in the more elementary topics, but it also touches on some advanced topics that he could well postpone until he could approach them in a different way. There is a good deal to interest the teacher of mathematics in the arrangement and presentation of the material to achieve the readable result. M. E. G.

Das mathematische Werkzeug des Chemikers, Biologen, Statistikers und Soziologen. Von R. FUETER. 3rd edition. Pp. 308. Brosch. Sw. fr. 14; geb. Sw. fr. 18.50. 1947. (Orell Füssli, Zürich)

The first edition was published in 1925 and grew out of lectures given by Professor Fueter in the University of Zürich, particularly in the faculty of medicine. The subject-matter consists of analytic geometry as far as the equations of the ellipse and hyperbola, differential calculus up to Taylor's theorem and partial derivatives, integral calculus with applications to areas and lengths of curves in cartesian or polar coordinates, easy differential equations of the first and second orders, an introduction to Fourier series, and a chapter of 46 pages on statistics in which there are five sections with the headings: (i) laws of probability, (ii) Gauss's error function, (iii) mathematical statistics, (iv) correlation, (v) significance test.

The presentation is elementary and clear, and the print is more openly spaced than we expect in these days. The book is thus easy to read, and it succeeds in its purpose of setting out the fundamentals of mathematics required by students of the natural sciences other than physicists and engineers. There are a number of examples, *e.g.* the equation of Van der Waals, damped oscillations, etc., chosen to enable the reader quickly to appreciate the ways in which the mathematical ideas may be applied, but little space is given to examples in which the interest is merely mathematical. For this reason, the book would no doubt serve a useful purpose in any language for the students for whom it is intended. It should be added, however, that it would hardly be sufficient for students who had examinations in mathematics ahead of them as there are no sets of exercises at the ends of the chapters, such as are customary in English textbooks. F. B.

Modern operational calculus. By N. W. MCLACHLAN. Pp. xiv, 218. 21s. 1948. (Macmillan)

Dr. McLachlan too modestly describes his book in the first line of the preface as "an introduction to *Modern Operational Calculus* based upon the Laplace transformation" even though it "is written for post-graduate engineers and technologists". Besides being post-graduate, in order to read

this book easily they would need to have had a previous introduction to operational methods and to such functions as Bessel functions and the error function, and to have some familiarity with the pure mathematics of the convergence of infinite series and integrals and, in one chapter, with complex integration.

Granted this background, the reader will find much of value and interest, not only in the application of the method to problems of engineering and technology that depend on ordinary and partial linear differential equations, but also to certain definite integrals and expansions (Chapter V). In fact, the seven chapters, covering 146 pages, give the reader in a compact form a sound impression of the directions in which the method has been applied and of the extent to which it has been successful. The rest of the book consists of six appendices (37 pages), mainly concerned with questions of mathematical rigour, a set of examples (24 pages) which " should be regarded as an integral part of the book ", containing other " important formulae and additional theorems ", a list of 81 Laplace transforms, ranging from the elementary functions to Bessel functions, Laguerre polynomials and Weber's parabolic cylinder function, and a short list of references. F. B.

Formeln und Sätze für die speziellen Funktionen der Mathematischen Physik. Von W. MAGNUS und F. OBERHETTINGER. Pp. viii, 230. 1948. Die Grundlehren der mathematischen Wissenschaften, 52; second edition (Springer, Berlin)

This extremely handy collection first appeared in 1943, so that copies were not available here. The new edition, we are told, not only corrects errors in the first, but adds a good deal of material. In particular, the section on elliptic functions has been re-written and doubled in size; perhaps E. H. Neville's *Jacobian elliptic functions* was not accessible to the authors, since they have not taken advantage of its systematising power.

The book sets out to list formulae and properties of the special functions of higher mathematics which are likely to be useful in mathematical physics. A list of the contents, by chapters, will show the scope: 1. The gamma function; 2. The hypergeometric function; 3. Cylinder functions (Bessel and the associates, Mathieu); 4. Spherical harmonics (mainly Legendre); 5. Orthogonal polynomials (Tschebyscheff, Hermite, Jacobi, Laguerre); 6. The confluent hypergeometric function, and its special cases; 7. Elliptic integrals, theta functions and elliptic functions; 8. Integral transforms and inversions (Fourier, Laplace, Hankel, Mellin); 9. Transformation of coordinates. There are appendices giving a number of Fourier expansions, and some summation formulae. A full bibliography would require at least as many pages as the whole book, but a five-page select list will serve for general purposes. Numerical tables are intentionally excluded, and the reader is sensibly referred to Rosenhead, Miller and Fletcher's *Index*. The reviewer has not attempted to check the whole enormous collection of formulae, but as far as he can tell from sampling there are very few misprints.

In view of the present interest in the transformation method of Laplace, and as an indication of the general scope of each section, here are some details about Chapter 8. The Fourier transform is defined, and 57 examples are given; then the Laplace transform is defined, its main rules of manipulation given by means of 16 formulae, and about 150 transforms listed; the Hankel and Mellin transforms follow, with 12 and 7 instances; there is a short note on the Gauss transform for a function of n variables; and finally, since these transforms are instances of integral equations with a singular kernel and an infinite range of integration, some corresponding examples with a finite range of integration are mentioned.

Any one, mathematician or mathematical physicist, who is interested in special functions will find this volume a most valuable addition to his shelves.

T. A. A. B.

The Differential Analyser. By J. CRANK. Pp. 137. 10s. 6d. 1947. (Longmans, Green)

Readers of the *Mathematical Gazette* have been aware of the existence of a machine for "The Mechanical Integration of Differential Equations" since Professor Hartree's paper under this title in vol. 22 (1938). This is still the best single account of the basic ideas and the performance of the differential analyser, which, to quote from the paper: "neither differentiates nor analyses, but, much more nearly, carries out the inverse of each of these operations". There exist two large-scale machines in Great Britain, namely at Manchester and at Cambridge, and in addition there have been built small-scale models, one of them at Macclesfield Grammar School. Such devices are, indeed, eminently suitable to illustrate the idea of those functional relationships defined by differential equations and the models should be not only instructive to students but also suggestive of didactic methods to teachers.

Differential analysers can now be considered to have passed through their experimental stage to maturity. A good deal of practical knowledge has been accumulated in operating them and the author should be congratulated on writing this readable account of his own experience and that of other workers in the field.

The first four chapters have the titles: I. Introduction, II. Construction of the Machine, III. Small-Scale Model Differential Analysers, IV. Designing a Simple Set-up. They explain how and why the machine works, what preparations are involved and what accuracy can be expected. The subsequent chapters enlarge on Hartree's 1938 paper in two respects. VI. Applications, and VII. The Mechanical Solution of Certain Types of Partial Differential Equations deal with these topics in welcome detail and make it clear how the appreciation of the physical significance of any mathematical problem can be a guide to the best method of its solution. Recent papers, including work done by the author, serve as illustrations. Moreover, Chapter VIII describes A New Differential Analyser, an account of which has been given by Bush and Caldwell in *J. Frank. Inst.*, vol. 240 (1945), and which incorporates radical departures from the original design. Two appendices are attached for the guidance of actual or would-be operators.

The fifth chapter is probably the most interesting for the reader mainly concerned with the ideas underlying machine solutions. It includes an account of methods for the generation of functions within the machine, by appropriate interconnections. This possibility is valuable, if input tables are to be avoided, and the idea is of mathematical interest. The most recent development of "regenerative connections", due to Amble in Oslo, is recorded. One would have liked to find, in addition to a table, small symbolic diagrams of the generations, as they were given in Appendix I to the paper by Bush and Caldwell mentioned above. This desire is made more acute by the pleasant and neat presentation of the existing diagrams and they would give, to the reviewer at any rate, more information than the printed word.

On p. 79 there is a figure which has been the victim of a curious mishap. It is quoted as being taken from *Proc. Phys. Soc.*, vol. 51, and a comparison shows that a mistake in the source (concerning the heading for integrator III) has been corrected. But worse confusion has been introduced by printing, on the margin, indices on the line instead of raising them. This will, no doubt, be corrected in a future edition and on that occasion the enumeration of the integrators in Fig. 20 might be brought into line with Fig. 19, which describes

precisely the same set-up. However, these are very minor flaws in an excellent presentation, which is interspersed with good photographic plates. Has the time come for monographs on some more calculating devices? They would be welcome. S. V.

Higher Algebra. By W. L. FERRAR. Pp. vi, 320. 17s. 6d. 1948. (Oxford University Press)

This is a sequel to the author's *Higher Algebra for Schools* and is intended for top mathematical forms in schools and first-year classes in universities. By a system of asterisks and other indications, the bookwork and examples are conveniently classified for ordinary sixth-form pupils, those approaching scholarship level, and first-year undergraduates.

The book begins with easy chapters (XV, XVI) on Finite and Infinite Series. There are then two chapters on Complex Numbers, with a theory starting from (r, θ) and the definition of a product, addition coming later. This removes the chief difficulty to the point where the laws of algebra are verified. The author recognises that this verification demands considerable detail, and he gives the proofs in outline. In the applications of complex numbers stress is laid on Argand Diagram examples, which have a good outlook value. In the worked examples of 7.4 the analytical method, using z and its conjugate $\bar{z}$, is preferred; but the useful result about inversion (often required for geometrical solutions) is given in 7.1.

Another interesting section is the work on Theory of Equations in Chapter XX (general) and Chapter XXI (selected topics). Good illustrations are given of the technique of finding " derived " equations. The selected topics include cubic and quartic equations, elimination, and equal roots.

There are chapters on difference equations and generating functions, partial fractions, inequalities, and continued fractions.

The general policy of the author is to explain methods, which he does very well, rather than to give formal proofs of general theorems. There are, however, a few of these proofs of special interest, *e.g.* a proof that a symmetric polynomial in the roots of an algebraic equation can be expressed in terms of the coefficients, using partitions; and the obtaining of an eliminant by a method which makes it clear that the condition is both necessary and sufficient.

The course provided by this book may be found not sufficiently ample for the best scholars. But it should be said that determinants and matrices have been deliberately omitted because the author feels that, after the stage of his more elementary book, these subjects should be studied in a separate text. The amount of ground to be covered at this stage depends also upon the views of the student and his teachers of the relative importance of algebra, analysis, geometry, and applied mathematics. A. R.

Analytische Geometrie. Von W. BLASCHKE. Pp. 152. DM. 10.50. 1948. (Wolfenbütteler Verlagsanstalt)

The series of which this is one volume is something of a makeshift. Germany, like the rest of Europe, is suffering from a severe shortage of scientific texts; the luxuries of good paper, stout binding, wide margins, must wait for better times. But if the format of the present volume is a little rough, the contents come from a masterly hand. Here—one is tempted to say, at last—is just the book for the bright sixth-former or first-year undergraduate, who wants to read a little geometry but is daunted by the austere algebraic abstractions on which modern algebraic geometry is built; here—again, one is tempted to say, at last—is a book in which algebra and geometry are not just added together, but fused into one entity, and in which results and for-

mulae are properly subordinated to methods. A long chapter at the end packs away most of the formulae where they can do no harm; the first hundred pages form the real book.

In the first chapter, we soon meet vectors and matrices, with examples from plane geometry, spherical trigonometry and kinematics to keep us in touch with the concrete. Chapter 2 is entitled "Spheres", but the survey is broad; the ideas of collineations and correlations appear, as do inversion, Laguerre's oriented circles, and the desmic configuration. Chapter 3 is about rotors and motors, with a flavour of statics and kinematics, Clifford's dual numbers, Study's *Übertragungsprinzip* and the Petersen-Morley theorem. A short chapter on moments of inertia introduces the characteristic equation of a symmetric matrix, the eigenvectors and principal axes, and thus prepares for Chapter 5 on quadrics. Chapter 6, the last of the main text, is on confocal quadrics and is again packed with interesting matter, Henrici's deformation theorem, Dupin's cyclides, geodesics on an ellipsoid, and so on.

To venture a comparison with Klein is to pay a compliment which Professor Blaschke deserves and will no doubt appreciate. T. A. A. B.

The Differential Geometry of Ruled Surfaces. By RAM BEHARI. Pp. 94. 1946. Lucknow University Studies, No. XVIII (Lucknow)

This monograph gives the substance of a series of Extension Lectures delivered by the author at the Lucknow University in 1942. It is concerned entirely with the classical theory, expressed in the notation of Forsyth, and its scope is indicated by the six chapter headings: Preliminary, Properties of the Line of Striction, Properties of the Generators and their Generalisations, Curved Asymptotic Lines of Ruled Surfaces, Deformation of Ruled Surfaces, and Ruled Surfaces of a Rectilinear Congruence. Much of the work in the last three chapters is the result of investigations made by the author. It is not quite clear for whom the book is intended, and in most universities it will fall between two stools. On the one hand it contains far more than can be included in the few lectures which are all that can be spared for ruled surfaces in a normal Honours course. To the specialist, on the other hand, it will not be of much use since it does not use the powerful and now common methods of vector algebra and tensor analysis. Many geometers will, however, add this to their collections of useful books of reference. A comprehensive bibliography is given. A. G. W.

Intermediate Commercial Arithmetic. By H. V. ALLEN. 2nd edition. Pp. viii, 269. 5s. 1947. (Longmans)

This work is based on the author's *Commercial Arithmetic* published in 1939 and now in process of being rewritten in two parts. The book under review is Part I and covers the requirements of the Intermediate Examinations of The Royal Society of Arts and of other Examining Bodies. The work will be completed later by Part II which will serve the needs of the advanced student. The subject-matter is taken almost entirely from the original work, which was reviewed in December 1939, but some of the explanations have been shortened and some slightly modified. Many of the sets of examples have been reduced in size but the examples are sufficient in number for a complete course. In the original review, the reviewer referred to the clear explanations, the careful development of the subject and the excellent and full descriptions of commercial practice. These qualities are just as evident in the new edition. An error in the old edition which has not been rectified occurs on page 42 where the Post Office interest on £20 for two months is given as 10d. instead of 1s. 8d.

S. I.

Mathematics for Modern Schools. Book 2. By T. H. WARD HILL. Pp. 312. 6s. 6d. 1947. (Harrap)

In Book 2 the author introduces Algebra and develops the other subjects further. In Arithmetic, Decimals are continued as far as multiplication and division. Proportion is introduced by the Unitary Method and there is an introduction to easy percentages. These subjects are developed by easy stages and difficulties are avoided. This method can be overdone as when division of decimals is confined almost entirely to examples which come out exactly. In practice, this type seldom occurs and much more emphasis should have been put on the division which does not come out exactly. A total of 4 short examples of this type and 2 more in later revision gives an unbalanced treatment of the subject.

The freshness of outlook which was characteristic of Book 1 is continued here and the author draws on applications to everyday life. Occasionally, however, he does not verify sufficiently some of the facts which he uses. For instance, he speaks of a summer holiday in Blackpool where the boarding-house charges are 3 guineas for adults and 2 guineas for children. Again, he states that Henry II created a new standard yard as equal to the length of his arm. He should have said Henry I. Incidentally, I have not come across any reliable metrologist who supports this statement. I have traced the story back as far as the *History* by Sir Richard Baker, 1643. But as this was written some 550 years after the supposed event, and as it is known to contain many inaccuracies, the event cannot be considered as more than apocryphal. All we know is that Henry I did create a standard yard (so did Edward I and Henry VII) and we must leave it at that.

Out of 12 chapters little more than one is devoted to Geometry. There is some revision and some compass work which includes bisection of lines and angles and the drawing of perpendiculars. The construction of triangles is also dealt with. As yet, there seems to be an absence of any clear aim in the Geometry.

Algebra, with Graphs, takes up 4 of the 12 chapters. The method of development is at a different tempo from the rest of the book where the reader is carried along in gentle short easy steps. This is the pace which the author considers as suitable for the average Modern School child, and I think he is right. In the Algebra section a much faster pace is set. Yet this is a much more difficult subject dealing, as it does, with generalities and abstractions. Perhaps the author is thinking here only of the top stream. How much Algebra, to whom to teach it, and how to present it are problems the solution of which must depend on future experiments. The author has given a working basis and that is perhaps all that we can expect for the present.

So far the author has taken up nearly 550 pages although he has not travelled very far. One wonders whether much of the explanation, helpful though it must be to many teachers, is not unnecessary in a pupils' book. S. I.

Modern Mathematics. By S. A. WALLING and J. C. HILL. Pp. 153. 3s. 6d. Notes and Answers, pp. 6 + 6 pp. answers, 1s. 6d. 1948. (Cambridge University Press)

The title of this book is unnecessarily misleading. The authors say that it is intended for the extra year of the Modern School. It will be clear from this that the book is not part of an integrated course, which is a pity, because the authors have something to say. In their remarks there is an economy of words used, yet the clarity of these remarks seems inversely proportional to the number of words used.

Teachers will find the development of the subject interesting. There are no sets of drill examples. All examples, with a few exceptions in logarithms,

are taken from industry and occupations. To help the teacher, each example is prefixed with a letter to indicate the application. A stands for agriculture, farming, etc. ; B is for building, surveying, construction ; Tr refers to transport (road, rail, sea, air) ; T has an application which is technical (mechanical, electrical, constructional), and G indicates a problem of general interest. Teachers will, therefore, find it easy to select or reject.

The authors urge the importance of accuracy and of checking answers. In Addition, instead of being asked to add a few numbers, the pupil is told that he is in charge of a motor van and he is asked to find the distance between Oxford and Bath from the given map. In another example, he is given similar instructions and he is asked to calculate the likely petrol consumption. He is given a dimensioned drawing of a machine part and is asked to calculate the thickness of a packing gland. Similar drawings are used for questions on Subtraction and Fractions.

In the chapter headed Proportion, two worked examples are given, each solved by both the unitary method and the multiplying factor method ; it is left to the teacher to select the method he prefers. The examples on graphs in a somewhat short chapter go straight to the point and all the examples indicate useful applications. The chapter on Mensuration includes the cylinder, cone, prism, pyramid, the sphere, the hipped roof, the truncated wedge and Simpson's Rule. Most of the formulae are given without any explanation of their validity, but, at any rate, they have the advantage of making the pupils familiar with the use of formulae and of appreciating their value. There is a further chapter on formulae which includes changing the subject. The authors are unduly optimistic if they think that pupils can master the matter in so short a time, particularly as some of the examples involve dividing both sides of an equation by a binomial expression.

The Geometry is confined to one chapter and again there is overcondensation and excess of optimism on the part of the authors. Another chapter deals with logarithms and the slide rule. The two parts of a circular slide rule are printed on thick pages. By cutting these out, pasting one part on a piece of wood and the other on a stout card, and putting a drawing pin through the centre, the pupil has the accurate equivalent of a 10″ slide rule. The rule has four positions which are marked Km., cm., ft., and gal. so that it can also be used as a ready reckoner for converting miles to Km., inches to cm., metres to feet and c. ft. to gallons. A graduation for π would have given added uses.

Mathematics teachers in Modern Schools will not find that this book will supply all their needs—far from it. Nevertheless, I advise them to get it and use it. If they do they will find themselves using it more and more and both they and their pupils will be stimulated by it.

There is a short chapter on Trigonometry and suggestions for the construction of simple surveying instruments, and at the end if the book there are tables of logarithms and trigonometrical ratios. S. I.

EUREKA

No. 11 of *Eureka* is in preparation and should appear shortly. Orders may be sent to the Editor, G. C. Shephard, Queens' College, Cambridge.

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A TREATISE

ON

CONIC SECTIONS,

CONTAINING

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THE MOST IMPORTANT MODERN ALGEBRAIC AND GEOMETRIC METHODS.

BY THE

REV. GEORGE SALMON,

FELLOW AND TUTOR, TRINITY COLLEGE, DUBLIN.

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